Thurs Feb 7

Summary:

The Lie groups $SO(3)$, $O(3)$, $SU(2)$ are 3-dimensional $\mathbb{R}$-manifolds with a group structure.

They are <u>locally</u> isomorphic but <u>globally</u> different.

Their tangent spaces (Lie algebras) are

$$\mathbb{R}^3 = \mathfrak{so}(3) = \mathfrak{o}(3) = \mathfrak{su}(2)$$

with the dot and cross product structures.

Topologically (i.e "globally") we have

$$\begin{tikzcd}
SU(2) = S^3 \arrow[dr, "2:1"'] & & O(3) = \underset{\det +1}{\mathbb{R}P^3} \sqcup \underset{\det -1}{\mathbb{R}P^3} \arrow[dl, "2:1"] \\
& SO(3) = \mathbb{R}P^3 &
\end{tikzcd}$$

Thus we can "lift" information from $SO(3)$ to $SU(2)$ and $O(3)$.

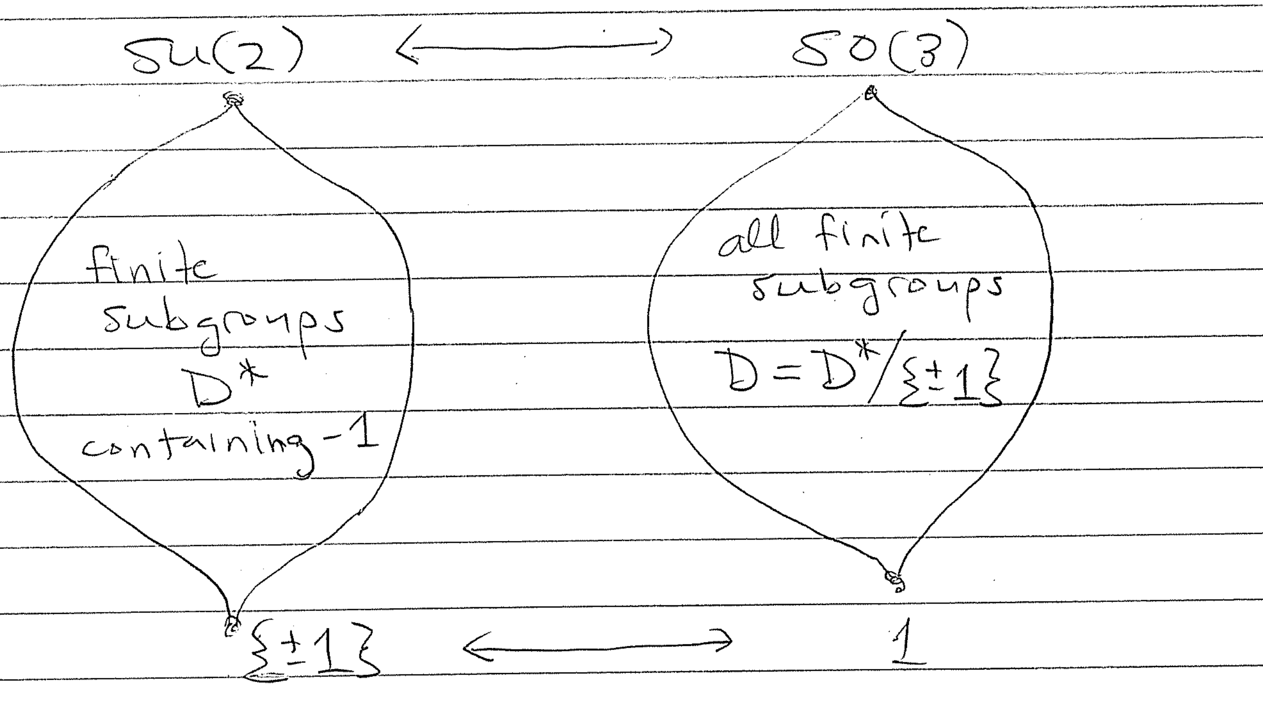
Start with $SU(2)$.
What information do we know about $SO(3)$?

Its finite subgroups are

$$D_{p,q,r} = \langle X^p = Y^q = Z^r = XYZ = 1 \rangle$$

for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

The spin homomorphism $SU(2) \longrightarrow SO(3)$ has kernel $\{\pm 1\}$ so we get a "correspondence":



Every polyhedral (von Dyck) group

$D_{p,q,r} < SO(3)$

lifts to a binary polyhedral group

$D^*_{p,q,r} < SU(2)$.

Facts:

- $D^*_{p,q,r} = \langle x^p = y^q = z^r = xyz \rangle$

- $|D^*_{p,q,r}| = 2\,|D_{p,q,r}|$

$$= \frac{4}{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1}$$

[Recall:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 + \frac{2}{|D_{p,q,r}|}.$$

]

- These are the only finite subgroups of $SU(2)$ that contain -1.

Q: Are there any other finite $G < SU(2)$?

Suppose $G < SU(2)$ is finite.

Two Cases:

(1) $|G|$ is even.

Then G contains an element of order 2. Indeed, break G into inverse pairs.

$$G = \bigcup_{g \in G} \{g, g^{-1}\}$$

Note: $|\{g, g^{-1}\}| = 1$ or 2 $\forall g \in G$, and $|\{1, 1^{-1}\}| = 1$.
Since $|G|$ is even, $\exists$ another element with $|\{g, g^{-1}\}| = 2$
$\implies g$ has order 2. ///

But $-1 \in SU(2)$ is the only element of order 2.

$\lightning$

Indeed, given $\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \in SU(2)$, suppose

$$\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}^2 = \begin{pmatrix} \alpha^2 - |\beta|^2 & \alpha\beta + \bar\alpha\beta \\ -\alpha\bar\beta - \bar\alpha\bar\beta & \bar\alpha^2 - |\beta|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\alpha^2 - |\beta|^2 = 1 = \bar\alpha^2 - |\beta|^2$

$$\Longrightarrow \quad \alpha^2 = \bar\alpha^2$$

$$\Longrightarrow \quad \alpha = \pm\bar\alpha$$

$$\Longrightarrow \quad \alpha \in \mathbb{R} \text{ or } i\mathbb{R}$$

If $\alpha = ix \in i\mathbb{R}$ then

$$1 = \alpha^2 - |\beta|^2 = -|x| - |\beta|^2 < 0 \quad \times$$

If $\alpha \in \mathbb{R}$ then $\alpha = \bar\alpha$

$$\Longrightarrow \quad 0 = \alpha\beta + \bar\alpha\beta = 2\alpha\beta \Longrightarrow \beta = 0$$

$$\Longrightarrow \quad 1 = \alpha^2 - |\beta|^2 = \alpha^2 \Longrightarrow \alpha = \pm 1$$

We conclude that $-1 \in G$, hence

$$G = D^*_{p,q,r}$$

(2) $|G|$ is odd.

If $|G|$ is odd then $-1 \notin G$ by Lagrange.
Restrict the spin homomorphism to G:

$$G \xrightarrow{\pi} SO(3).$$

Since $\ker \pi = G \cap \{\pm 1\} = 1$ we have

$$G \approx \pi(G) < SO(3).$$

Hence $G \approx D_{p,q,r}$.

Then $|D_{p,q,r}| = \dfrac{2}{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1} = \text{odd}.$

$\Longrightarrow$ $p,q,r = 1,1,n$ for odd n.

$\Longrightarrow$ G is odd and cyclic

$\blacksquare$

We have proved ...

<u>Theorem</u>: The finite subgroups of $SU(2)$ are

- cyclic C_n^* and odd cyclic
- binary dihedral D_{2n}^* $4n$
  ("dicyclic")
- binary tetrahedral T^* 24
- binary octahedral O^* 48
- binary icosahedral I^* 120

That's All.

[Remark: In fact, any finite subgroup to $SL_2(\mathbb{C})$ is conjugate to one of these.

Any <u>compact</u> subgroup of $SL_n(\mathbb{C})$ is conjugate to a unitary group $< U(n)$.
]

Finally: McKay Correspondence.

In 1980, John McKay noticed something strange...

There are two kinds of things parametrized by triples $(p,q,r) \in \mathbb{N}^3$ with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

Graphs with spectral radius 2		Finite subgroups of $SU(2)$.
$\hat{A}_n$		Cyclic
$\hat{D}_n$		dicyclic
$\hat{E}_6$	$\longleftrightarrow$	T^*
$\hat{E}_7$		O^*
$\hat{E}_8$		I^*

Does this mean anything?

McKay: Yes!

Crash course on representation theory:

Given finite abstract group G, a representation is a group hom

$$\rho : G \longrightarrow GL_n(\mathbb{C}) = \mathrm{End}_{\mathbb{C}}(V)$$

Representations can be "added" and "multiplied".

Given $\rho_1 : G \to \mathrm{End}(V)$
$\rho_2 : G \longrightarrow \mathrm{End}(W)$, we get

$$\rho_1 \oplus \rho_2 : G \longrightarrow \mathrm{End}(V \oplus W)$$
$$\rho_1 \otimes \rho_2 : G \longrightarrow \mathrm{End}(V \otimes W).$$

We say $\rho : G \to \mathrm{End}(V)$ is irreducible if it cannot be written as a sum.

Theorem (Frobenius, ~1896):

G has finitely many irr. reps.

irr. reps. = # conj. classes in G.

However there is no canonical bijection.

Examples:

- The "trivial" representation

$$\begin{aligned} \text{triv}: G &\longrightarrow GL_1(\mathbb{C}). \\ g &\longmapsto \text{id}. \end{aligned}$$

is irreducible.

- The "regular" representation

$$\text{reg}: G \longrightarrow GL_{|G|}(\mathbb{C}).$$

Consider the formal vector space

$$\mathbb{C}[G] = \left\{ c_1\bar{g}_1 + c_2\bar{g}_2 + \cdots + c_n\bar{g}_n : \begin{array}{l} \text{where } c_1, c_2, \ldots, c_n \in \mathbb{C}. \\ \text{and } \{g_1, g_2, \ldots, g_n\} = G \end{array} \right\}.$$

Then G acts on $\mathbb{C}[G]$ by

$g \cdot \bar{h} := \overline{gh}$ (extend linearly).

Theorem (Frobenius, ~1896):

Let $W^{(1)}, W^{(2)}, \ldots, W^{(k)}$ be all the irreps of G. Then

$$\mathbb{C}[G] = \bigoplus_{i=1}^{k} m_i W^{(i)}$$

where $m_i = \dim W^{(i)}$.

Corollary: $\sum_{i=1}^{k} m_i^2 = \dim \mathbb{C}[G]$
$= |G|$.

We can simplify a representation by taking its "character":

$$\chi_\rho : G \longrightarrow \mathbb{C}.$$

$$\begin{tikzcd} G \arrow[r, "\rho"'] \arrow[rr, bend right, "\chi_\rho"'] & GL_n(\mathbb{C}) \arrow[r, "\text{trace}"'] & \mathbb{C}. \end{tikzcd}$$

Miracle (Frobenius):
χ_ρ determines ρ up to isomorphism!