Thurs Jan 31

Theorem (Samuelson, 1940):
Only three spheres admit a topological group structure:

$$S^0 = O(1) = \{\pm 1\} = \{x \in \mathbb{R} : |x| = 1\}$$

$$S^1 = SO(2) = U(1) = \{z \in \mathbb{C} : |z| = 1\}$$

$$S^3 = SU(2) = Sp(1) = \{q \in \mathbb{H} : |q| = 1\}.$$

That's All!

Today we will discuss S^3.

Recall the algebra of quaternions

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

with "absolute value" $|q|^2 = \det(q)$

and "conjugation" $q^* = \bar{q}^t$

Note that

$$qq^* = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\bar{\alpha} + \beta\bar{\beta} & 0 \\ 0 & \alpha\bar{\alpha} + \beta\bar{\beta} \end{pmatrix}$$

$$= (\alpha\bar{\alpha} + \beta\bar{\beta}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= |q|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

As we saw, this implies that the group of <u>unit</u> quaternions is

$$Sp(1) = SU(2)$$

Recall the $\mathbb{R}$eal structure of $\mathbb{H}$

$$a\hat{1} + b\hat{i} + c\hat{j} + d\hat{k} =$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} + d\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

So $\mathbb{H} = \{ a\hat{1} + b\hat{i} + c\hat{j} + d\hat{k} : a, b, c, d \in \mathbb{R} \}$
$= \mathbb{R}^4$

with relations

$$\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = \hat{i}\hat{j}\hat{k} = -\hat{1}$$

Note that

$$|a\hat{1} + b\hat{i} + c\hat{j} + d\hat{k}| = a^2 + b^2 + c^2 + d^2$$
$$= \|(a, b, c, d)\|^2_{\mathbb{R}^4}$$

$\Longrightarrow \mathbb{H} = \mathbb{R}^4$ geometrically.

Note that $Sp(1) = SU(2)$ acts on $\mathbb{H} = \mathbb{R}^4$ by isometries:

Given $u \in Sp(1)$, let $\mathbb{H} \xrightarrow{u} \mathbb{H}$
$$q \longmapsto u^{-1} q u.$$

Then $|u^{-1}pu - u^{-1}qu| = |u^{-1}(p-q)u|$

$$= \underbrace{|u^{-1}|}_{1} \cdot |p-q| \cdot \underbrace{|u|}_{1} = |p-q|.$$

$\blacksquare$

We get a group homomorphism

$$Sp(1) \longrightarrow O(4).$$

Q: Surjective, Injective?
NO NO.

More subtly, $Sp(1)$ acts on $\mathbb{R}^3$ by isometries.

Think $\mathbb{H} = \mathbb{R}^4 = \mathbb{R} \oplus \mathbb{R}^3$

$$= \langle \hat{1} \rangle \oplus \underbrace{\langle \hat{i}, \hat{j}, \hat{k} \rangle}_{\text{Imaginary quaternions}}$$

We identify

$$\mathbb{R}^3 = \{ai + bj + ck : a, b, c \in \mathbb{R}\}$$
$$= \mathbb{R}^{\perp} = \mathbb{R}\hat{1}^{\perp}$$

Note the action $Sp(1) \curvearrowright \mathbb{R}^4$ stabilizes the scalar subspace

$t \in Sp(1), r \in \mathbb{R} \Rightarrow t^{-1}rt = rt^{-1}t = r.$

Since $Sp(1)$ acts by isometries, it also stabilizes the orthogonal complement $\mathbb{R}^3 = \mathbb{R}^{\perp}$.

We get a group homomorphism

$$Sp(1) \longrightarrow O(3).$$

Describe it!

First note that $\mathbb{R}^3$ is NOT closed under multiplication. Given $\vec{u}, \vec{v} \in \mathbb{R}^3$

$$\vec{u} = u_1 i + u_2 j + u_3 k$$
$$\vec{v} = v_1 i + v_2 j + v_3 k, \quad \text{we have}$$

$$\vec{u}\vec{v} = -u_1v_1 + u_1v_2 k - u_1v_3 j$$
$$- u_2v_1 k - u_2v_2 + u_2v_3 i$$
$$+ u_3v_1 j - u_3v_2 i - u_3v_3$$

$$= -(u_1v_1 + u_2v_2 + u_3v_3)$$

$$+ \left[(u_2v_3 - u_3v_2) i + (u_3v_1 - u_1v_3) j + (u_1v_2 - u_2v_1) k\right]$$

$$= -\vec{u}\cdot\vec{v} + \vec{u}\times\vec{v}$$

scalar part vector part.

[Remark: This is the origin of dot and cross product. Before $\mathbb{R}^3$ was a "vector space", it was the space of imaginary quaternions.]

Corollary: For all $\vec{u} \in \mathbb{R}^3$ we have

$$\vec{u}^2 = \vec{u}\vec{u} = -\vec{u}\cdot\vec{u} + \underbrace{\cancel{\vec{u}\times\vec{u}}}_{0}$$

$$= -\|\vec{u}\|^2 \in \mathbb{R}$$

If $\vec{u} \in \mathbb{R}^3 \cap Sp(1)$

$$\vec{u}^2 = -\|\vec{u}\|^2 = -1$$

[Remark: If $\mathbb{H}$ were a field, the equation $x^2+1=0$ would have ≤ 2 solutions. Instead it has uncountably many:

$$\mathbb{R}^3 \cap Sp(1) = S^2$$]

The Polar Form of $Sp(1)$:

Given $t \in Sp(1)$ let

$$t = t_0 + \vec{t}$$
$$= a1 + (bi + cj + dk)$$

Since
$$|t|^2 = a^2 + b^2 + c^2 + d^2$$
$$= t_0^2 + \|\vec{t}\|^2 = 1$$

we conclude that

$(t_0, \|\vec{t}\|) = (\cos\theta, \sin\theta)$ for some $\theta \in \mathbb{R}$.

and hence $t = \cos\theta + \dfrac{\vec{t}}{\|\vec{t}\|} \sin\theta$.

In other words, every $t \in Sp(1)$ has a polar form

$$\boxed{t = \cos\theta + \vec{u} \sin\theta}$$

where $\vec{u} \in \mathbb{R}^3$ and $\|\vec{u}\| = 1$.
(and hence also $\vec{u}^2 = -1$).

Finally we have

Theorem: If $t = \cos\theta + \vec{u}\sin\theta \in Sp(1)$, then the isometry $\mathbb{R}^3 \to \mathbb{R}^3$ defined by $\vec{v} \longmapsto t^{-1}\vec{v}\,t$ is just

$$\boxed{\text{Rotation around axis } \vec{u} \text{ by } 2\theta}$$

Proof: First observe that t fixes the line $\mathbb{R}\vec{u} \subseteq \mathbb{R}^3$:

$$t^{-1}\vec{u}\,t = (\cos\theta - \vec{u}\sin\theta)\,\vec{u}\,(\cos\theta + \vec{u}\sin\theta)$$

$$= (\vec{u}\cos\theta - \overset{-1}{\cancel{\vec{u}^2}}\sin\theta)(\cos\theta + \vec{u}\sin\theta)$$

$$= \vec{u}(\cos^2\theta + \sin^2\theta) + \cancel{\sin\theta\cos\theta} + \overset{-1}{\cancel{\vec{u}^2}}\cancel{\sin\theta\cos\theta}$$

$$= \vec{u} \quad ///$$

Now let $\{\vec{u}, \vec{v}, \vec{w}\}$ be an orthonormal basis for $\mathbb{R}^3$ with $\vec{w} = \vec{u} \times \vec{v}$. Since $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$, we have

$$\vec{u}^2 = \vec{v}^2 = \vec{w}^2 = \vec{u}\vec{v}\vec{w} = -1$$

We claim that t acts on the plane

$$\{r_1\vec{v} + r_2\vec{w} : r_1, r_2 \in \mathbb{R}\} = \mathbb{R}\vec{u}^{\perp}$$

by $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$.

Enough to check the basis $\vec{v}, \vec{w}$:

$$t^{-1}\vec{v}\,t = (\cos\theta - \vec{u}\sin\theta)\,\vec{v}\,(\cos\theta + \vec{u}\sin\theta)$$

$$= (\vec{v}\cos\theta - \vec{u}\vec{v}\sin\theta)(\cos\theta + \vec{u}\sin\theta)$$

$$= \vec{v}\cos^2\theta - \vec{u}\vec{v}\sin\theta\cos\theta + \overset{-\vec{u}\vec{v}}{\cancel{\vec{v}\vec{u}}}\sin\theta\cos\theta - \vec{u}\,\overset{-\vec{u}\vec{v}}{\cancel{\vec{v}\vec{u}}}\sin^2\theta$$

$$= \vec{v}\cos^2\theta - 2\overset{\vec{w}}{\cancel{\vec{u}\vec{v}}}\sin\theta\cos\theta + \overset{-1}{\cancel{\vec{u}^2}}\vec{v}\sin^2\theta.$$

$$= \vec{v}(\cos^2\theta - \sin^2\theta) - 2\vec{w}\sin\theta$$

$$= \vec{v}\cos 2\theta - \vec{w}\sin 2\theta$$

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Similarly,

$$t^{-1}\vec{w}\,t = \vec{v}\sin 2\theta + \vec{w}\cos 2\theta.$$

∎

This action gives a surjective hom.

$$Sp(1) \twoheadrightarrow SO(3)$$

Kernel?

Note that rotation around $\vec{u}$ by 2θ corresponds to

$$t = \cos\theta + \vec{u}\sin\theta$$

and $\cos(\theta+\pi) + \vec{u}\sin(\theta+\pi)$

$$= -\cos\theta - \vec{u}\sin\theta$$
$$= -t$$

and no other element of $Sp(1)$.

(Easy to see $(-t)^{-1}\vec{v}(-t) = t^{-1}\vec{v}t$.)

Hence the kernel is $\{\pm 1\}$.

$\Longrightarrow$

$$SO(3) \approx \frac{Sp(1)}{\{\pm 1\}} = \frac{S^3}{\{\pm 1\}} = \mathbb{R}P^3.$$

identify antipodal points