Thurs Jan. 24

Consider a "regular" polyhedron $Q \subseteq \mathbb{R}^3$

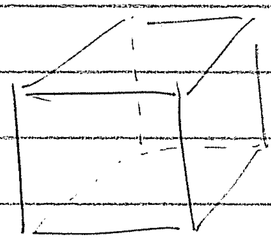
i.e. $G := \mathrm{Aut}^+(Q) \leq SO(3)$ acts <u>transitively</u> on vertices / edges / faces.

Let v, e, f = # vertices, edges, faces of Q

Euler says: $v - e + f = 2$ (*)
"Euler characteristic of S^2"

Let r_v, r_e, r_f = # $g \in \mathrm{Aut}^+(Q)$ fixing a given vertex, edge, face.

eg



$v = 8$ $\quad r_v = 3$
$e = 12$ $\quad r_e = 2$
$f = 6$ $\quad r_f = 4$

Orbit-Stabilizer says:

$$r_v v = r_e e = r_f f = |G|.$$

[Note: we always have $r_e = 2$

$$\Rightarrow e = \frac{|G|}{2}$$]

By double-counting the set

$\{(\text{vertex}, \text{edge}) : \text{incident}\}$

we get $r_v v = 2e \Rightarrow v = \frac{2e}{r_v}$

Similarly, double-counting the set

$\{(\text{face}, \text{edge}) : \text{incident}\}$

gives $r_f f = 2e \Rightarrow f = \frac{2e}{r_f}$.

Substitute into Euler (*) :

$$v - e + f = 2.$$

$$\frac{2e}{r_v} - e + \frac{2e}{r_f} = 2.$$

$$\frac{1}{r_v} - \frac{1}{2} + \frac{1}{r_f} = \frac{1}{e} = \frac{2}{|G|}$$

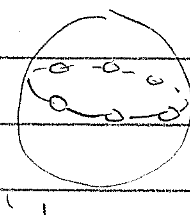
$$\frac{1}{r_v} + \frac{1}{r_f} = \frac{2}{|G|} + \frac{1}{2}$$

Finally, add $\frac{1}{r_e} = \frac{1}{2}$ to both sides

$$\boxed{\frac{1}{r_v} + \frac{1}{r_e} + \frac{1}{r_f} = 1 + \frac{2}{|G|}}$$

Hence the regular polyhedra are :

Polygon (2,2,a)



Tetrahedron (3,2,3)

"degenerate"

Cube (3,2,4)
Octahedron (4,2,3)

Icosahedron (5,2,3)
Dodecahedron (3,2,5).

That's All !

Conversely, let $G < SO(3)$ be <u>finite</u> and consider $G \curvearrowright S^2 \subseteq \mathbb{R}^3$

Let $P := \{ p \in S^2 : \exists\, 1 \neq g \in G \text{ with } g(p) = p \}$

"poles".

Lemma: $G \curvearrowright P$ ▨

Suppose P breaks into m G-orbits with stabilizers of size $r_1, r_2, \dots, r_m$.

Then ... $\boxed{2 - \frac{2}{|G|} = \sum_{i=1}^{m} \left(1 - \frac{1}{r_i}\right)}$

$m = 1$ and $m \geq 4$ are impossible. So there are two cases.

$m = 2$:

$(r_1, r_2) = (|G|, |G|) \Rightarrow G$ is cyclic.

$m = 3$: $(r_1, r_2, r_3) = (r_v, r_e, r_f)$

$(2, 2, a) \Rightarrow G = D_{2a}$
(Exercise)

$(2, 3, 3) \Rightarrow G = \mathrm{Aut}^+(\text{Tetra.})$

$(2, 3, 4) \Rightarrow G = \mathrm{Aut}^+(\text{Cube/Oct.})$

$(2, 3, 5) \Rightarrow G = \mathrm{Aut}^+(\text{Icos./Dodec.})$

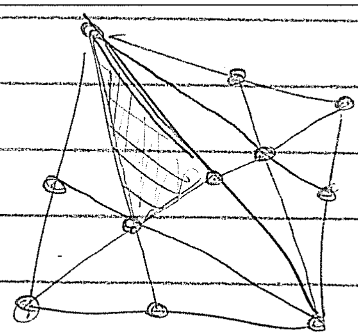
That's all!

We have classified finite/discrete subgroups of $SO(3)$

Group Presentation?

Consider the barycentric subdivision $\Delta(Q)$ of Q.

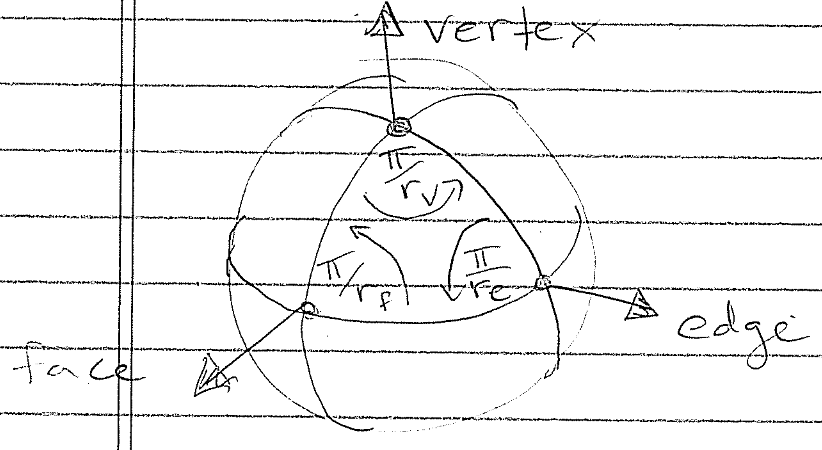
Eg



Δ(Tetrahedron).

Faces of $\Delta(Q)$ are "flags" (vertex $\subseteq$ edge $\subseteq$ face) in Q.

Project $\Delta(Q)$ onto the sphere S^2



See Handout.

Let $A_v, A_e, A_f \in SO(3)$ be rotations around v, e, f by $\frac{2\pi}{r_v}, \frac{2\pi}{r_e}, \frac{2\pi}{r_f}$, c.c.w.

One can show that:

1) Given <u>any</u> flag $(v \subseteq e \subseteq f)$, the group $Aut^+(Q)$ is generated by A_v, A_e, A_f.

2) The <u>only</u> relations are

$$A_v^{r_v} = A_e^{r_e} = A_f^{r_f} = \underbrace{A_v A_e A_f}_{\text{(Euler)}} = 1.$$

Thus one can <u>abstractly</u> define the polyhedral groups

$$P_{p,q,r} := \langle X, Y, Z : X^p = Y^q = Z^r = XYZ = 1 \rangle$$

Degenerate Case:

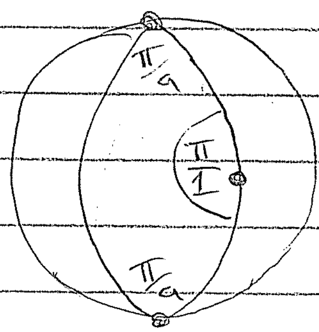
$$P_{1,a,a} = \langle X^1 = Y^a = Z^a = XYZ = 1 \rangle$$

$$X^1 = 1 \implies X = 1$$
$$YZ = 1 \implies Y = Z^{-1}.$$

$$\implies P_{,a,a} = \langle Y^a = 1 \rangle \text{ Cyclic.}$$

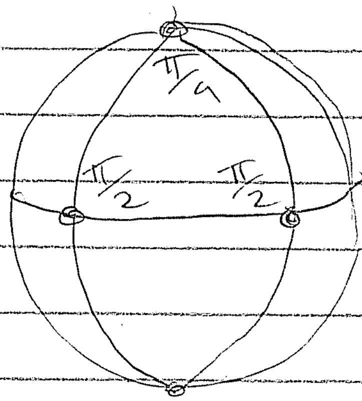
Picture:

Cyclic $P_{1,a,a}$



"Degenerate triangle" = "digon"

Dihedral $P_{2,2,a}$

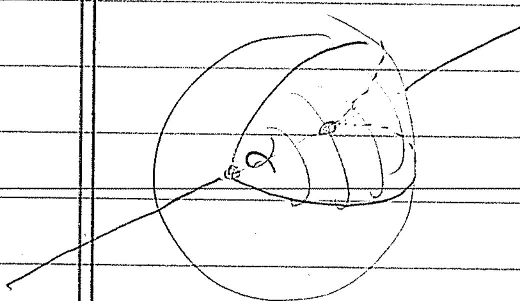


Half a digon.

Finally, compute the area of a triangle on the sphere of radius R.

<u>Lemma 1</u>: Area of the sphere $= 4\pi R^2$.

<u>Lemma 2</u>: Area of digon with angle α



$$= 4\pi R^2 \left(\frac{\alpha}{2\pi}\right)$$

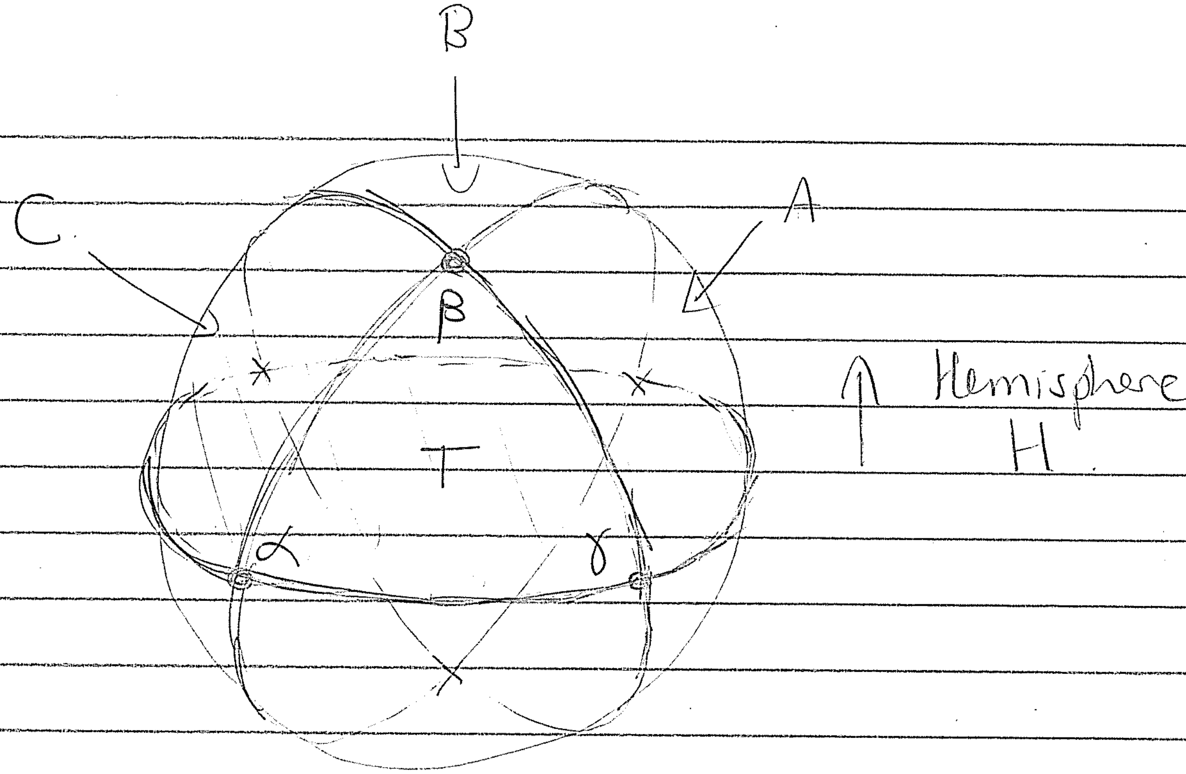
$$= 2\alpha R^2.$$

Theorem: Area of triangle with angles α, β, γ is

$$\underbrace{(\alpha + \beta + \gamma - \pi)}_{\text{"angle excess"}} R^2$$

Proof (Thomas Harriot, 1603):

Extend the edges of the triangle:



Let σ mean area: $\sigma(H) = 2\pi R$.

$$H = T \sqcup A \sqcup B \sqcup C$$

Note
$$\left.\begin{aligned} \sigma(A) + \sigma(T) &= 2\alpha R^2 \\ \sigma(B) + \sigma(T) &= 2\beta R^2 \\ \sigma(C) + \sigma(T) &= 2\gamma R^2 \end{aligned}\right\} \text{digons.}$$

Hence
$$\begin{aligned} 2\pi R^2 &= \sigma(H) \\ &= \sigma(T) + \sigma(A) + \sigma(B) + \sigma(C) \\ &= \sigma(T) + (2\alpha R^2 - \sigma(T)) + (2\beta R^2 - \sigma(T)) + (2\gamma R^2 - \sigma(T)) \\ &= 2R^2(\alpha + \beta + \gamma) - 2\sigma(T). \end{aligned}$$

$$\Longrightarrow \quad \sigma(T) = (\alpha + \beta + \gamma - \pi) R^2.$$

$\blacksquare$

Note $\sigma(T) > 0$!

In particular, if $(\alpha, \beta, \gamma) = \left(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}\right)$

Then
$$\left(\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} - \pi\right) R > 0$$

$$\Longrightarrow \boxed{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1} \quad (*)$$

Again!

Conclusion:

$(*) \Longleftrightarrow \exists$ spherical triangle with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$.

Q: What can one say about the abstract group

$$P_{p,q,r} = \langle X^p = Y^q = Z^r = XYZ = 1 \rangle$$

when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$??

(Assignment: Explore "KaleidoTile")