Thurs Apr 25

Final Lecture: Numerology

Let G be a FGGR with simple roots $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Assume $\|\alpha_i\|^2 = 2$ for all i and consider $A = (\alpha_1 \; \alpha_2 \cdots \alpha_n)$. We know that the Coxeter adjacency matrix

$$2I - A^t A$$

has eigenvalues < 2. Specifically they are

$$2\cos\left(\frac{(d_1-1)\pi}{h}\right), 2\cos\left(\frac{(d_2-1)\pi}{h}\right), \ldots, 2\cos\left(\frac{(d_n-1)\pi}{h}\right)$$

for some God-given integers called the "degrees" of G,

$$d_1 \le d_2 \le \cdots \le d_n =: h.$$

The largest degree $d_n =: h$ is called the "Coxeter number" of G.

Without further ado, here they are:

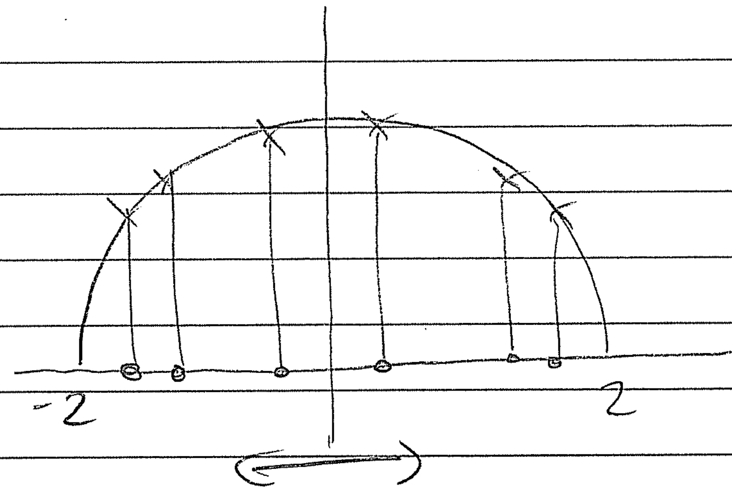
G	Degrees	h
A_n	$2, 3, 4, \dots, n+1$	$n+1$
BC_n	$2, 4, 6, \dots, 2n$	$2n$
D_n	$2, 4, 6, \dots, 2n-2, n$	$2n-2$
E_6	$2, 5, 6, 8, 9, 12$	12
E_7	$2, 6, 8, 10, 12, 14, 18$	18
E_8	$2, 8, 12, 14, 18, 20, 24, 30$	30
F_4	$2, 6, 8, 12$	12
$G_2(m)$	$2, m$	m
H_3	$2, 6, 10$	10
H_4	$2, 12, 20, 30$	30

The degrees have AMAZING properties:

- Since the Cox. diagram is bipartite, the spectrum is symmetric about 0.

$$\Longrightarrow \underbrace{d_i + d_{n-i+1} = h+2}_{\text{symmetry.}} \quad \forall i.$$

Picture:



- The size of the group is

$$|G| = d_1 d_2 \cdots d_n.$$

Eg $|\mathfrak{S}_n| = 2 \cdot 3 \cdot 4 \cdots n = n!$

$|\text{type } BC_n| = 2 \cdot 4 \cdot 6 \cdots 2n = 2^n \cdot n!$

$|\text{type } H_3| = 2 \cdot 6 \cdot 10 = 120$

$|\text{type } E_8| = 2 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \cdot 20 \cdot 24 \cdot 30$

$= 696{,}729{,}600$ elements

This can be refined in two ways:

Let S = simple reflections
T = all reflections $= GSG^{-1}$.

Define length functions $l_S, l_T : G \to \mathbb{N}$.

$l_{S/T}(g) :=$ min k such that g is a product of k elements of S/T.

l_S = S-word length
l_T = T-word length

Define the q-integer: for all $N \in \mathbb{N}$, let $[N]_q = 1 + q + q^2 + \cdots + q^{N-1}$.

Then we define two "q-analogues" of the number $|G|$.

(1) Let $G_S(q) = \sum_{g \in G} q^{l_S(g)}$.

Theorem: $G_S(q) = [d_1]_q [d_2]_q \cdots [d_n]_q$

Example:

$\mathfrak{S}_{n\,S}(q) = [2]_q [3]_q \cdots [n]_q = [n]_q!$

Note: $\ell_S(g)$ = # hyperplanes separating g from 1.
= # "inversions" of g.

② Let $G_T(q) = \sum_{g \in G} q^{\ell_T(g)}$

Theorem: $G_T(q) = \prod_{i=1}^{n} (1 + (d_i - 1)q)$

Note $G_T(1) = \prod (1 + (d_i - 1)) = \prod d_i = |G|$.

But we get more.

The number of reflections $|T|$ is the coefficient of q in $G_T(q)$. Hence

$$|T| = \frac{d}{dq} G_T(q) \Big|_{q=0} = \sum_{i=1}^{n} (d_i - 1)$$

In fact we have

$$\ell_T(g) = n - \dim(\ker(g-1)).$$

$$\ell_T(\text{reflection}) = n - (n-1) = 1.$$

The number of $g \in G$ with fixed space of dim $n-k$ is the k^{th} elementary symm. combination of

$$d_1 - 1, d_2 - 1, \ldots, d_n - 1.$$

Eg The # elements with trivial fixed space

$$= \prod_{i=1}^{n} (d_i - 1)$$

Eg. In $\mathfrak{S}_n$ these are exactly the n-cycles. The # n-cycles is

$$\prod (d_i - 1) = 1 \cdot 2 \cdot 3 \cdots (n-1) = (n-1)!$$

This generalizes:

Definition: Let $S = \{s_1, s_2, \ldots, s_n\}$. The product $s_{\mu(1)} s_{\mu(2)} \cdots s_{\mu(n)}$ for any $\mu \in \mathfrak{S}_n$ is called a "Coxeter element" of G.

Fact: Coxeter elements are conjugate. Let $\mathcal{C}$ denote the conjugacy class of Coxeter elements.

Think Cox. elt = "n-cycle"

Theorems: Consider any $c \in \mathcal{C}$.

- $|\langle c \rangle| = h = d_n$, the Coxeter number.
- $Z(c) = \langle c \rangle$, hence

$$|\mathcal{C}| = |G| / |Z(c)| = \frac{d_1 \cdot d_2 \cdots d_n}{d_n}$$

$$= d_1 \cdot d_2 \cdots d_{n-1}$$

- $\ker(c-1) = \{0\}$, hence $\mathcal{C}$ are among the $\prod_{i=1}^{n} (d_i - 1)$ elements

with trivial fixed space..

Corollary:

$$d_1 d_2 \cdots d_{n-1} \leq \prod_{i=1}^{n} (d_i - 1)$$

Fact: Equality holds $\iff G = \mathfrak{S}_n$

$$2 \cdot 3 \cdots (n-1) = (2-1)(3-1) \cdots (n-1).$$

- c stabilizes a special 2-dim plane called the "Coxeter plane".

Cox. diagram is a tree $\Longrightarrow$ we can bicolor the vertices

$$S = L \sqcup R$$

such that $s \in L$ all commute
$s \in R$ all commute.

Let $\ell = \prod_{s \in L} s$, $r = \prod_{s \in R} s$.

and consider Coxeter elt $c = \ell r$.

Note: ℓ, r are involutions

Fact: ℓ, r restricted to the Coxeter plane are reflections that generate a dihedral group of order $2h$

$c = \ell r$ acts as rotation by $2\pi/h$ on the Coxeter plane.

- we know that all $c \in \mathcal{C}$ have the same eigenvalues, which are hth roots of unity.

<u>Theorem</u> (Coxeter). Let $\omega = e^{2\pi i/h}$.
The eigenvalues of c are

$$\omega^{d_1-1}, \omega^{d_2-1}, \ldots, \omega^{d_n-1}$$

Hence the numbers $d_1-1, \ldots, d_n-1$ are called the "exponents" of G.

$\Rightarrow$ Gives a new proof of symmetry.
Eigenvalues come in conjugate pairs.

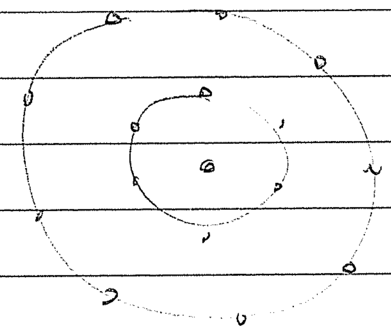
Hence $\omega^{d_i-1}\omega^{d_{n-i+1}-1} = 1$.

$$\Rightarrow (d_i-1)+(d_{n-i+1}-1) = h \quad \forall i \quad \checkmark.$$

Corollary: # reflections $= |T| = \dfrac{nh}{2}$.

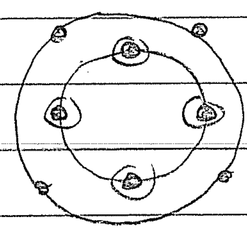
<u>Proof</u>: $|T| = \sum(d_i-1)$
$= \sum[h-(d_i-1)]$
$= nh - \sum(d_i-1)$
$= nh - |T|$. ∎

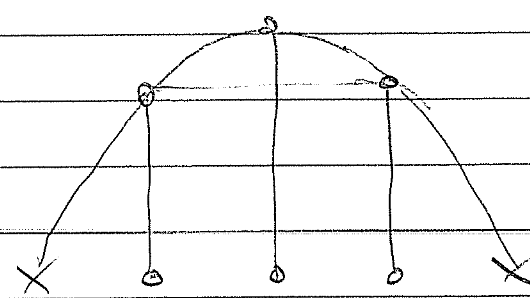
Now project the root system onto the Coxeter plane. The roots fall into concentric circles:



Theorem: The radii of the circles are the entries of the PF eigenvector for eigenvalue $2\cos\left(\frac{(h-1)\pi}{h}\right) < 2$

Eg Type A_3.





Eg Recall E_8 and the 1-dim magnet

Story: Coxeter "discovered" the degrees while watching Chevalley at the ICM in 1950.

Chevalley was presenting a result:

Let G be crystallographic with compact Lie group $\mathrm{Lie}(G)$.
The Poincaré series encodes the Betti numbers

$$P(X,q) = \sum_{i \geq 0} \dim H_i(X,\mathbb{R}) \cdot q^i$$

Chevalley observed that

$$P(\mathrm{Lie}(G),q) = \prod_{i=1}^{n} (1+q^{2d_i - 1}). \quad (*)$$

Coxeter immediately recognized these d_i.

<u>Problem</u>: When G is not crystallographic, i.e. $\mathrm{Lie}(G)$ does not exist, is $(*)$ the Poincaré series of something?

Closely related result. Consider the complexified complement of the reflecting hyperplanes

$$\mathbb{C}^n \setminus \bigcup_{\text{real } H \in \Sigma} H . \quad \left(\begin{array}{c}\text{not} \\ \text{disconnected}\end{array}\right)$$

Theorem (Orlik-Solomon): Its Poincaré series is

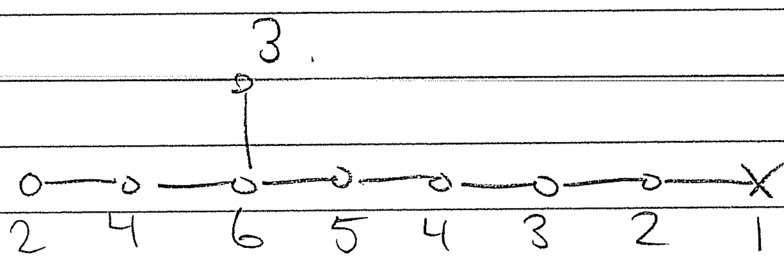
$$G_T(q) = \prod_{i=1}^{n} \left(1 + (d_i - 1) q \right)$$

[Related to the Weil conjectures.]

Affine Things.

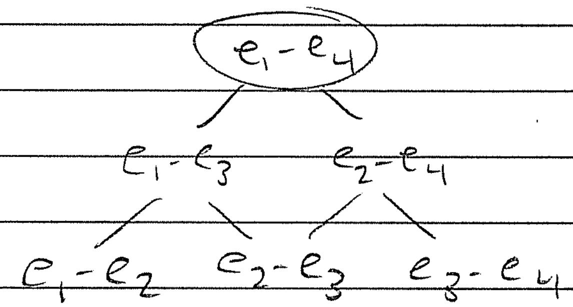
Consider the PF eigenvector of an affine Coxeter diagram

Eg $E_8^{(1)}$

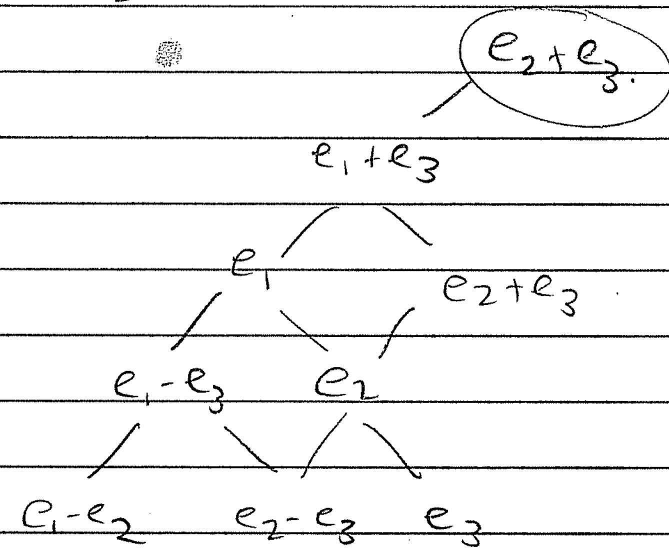


We can partially order the positive roots Φ^+ by $\alpha \leq \beta \iff \beta - \alpha \in \mathbb{Z}^+\Pi$.

Eg.



A_3



B_3.

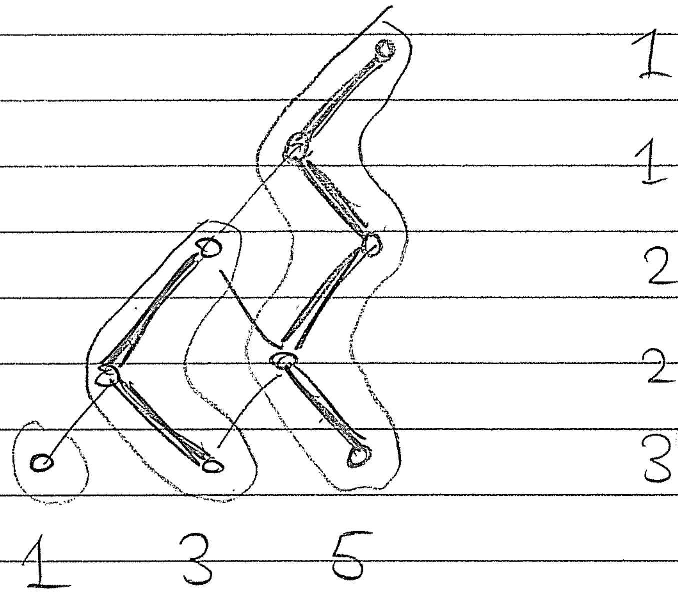
Let $ht(\alpha)$ be the height of α in this poset.

So $ht(\alpha) = 1 \iff \alpha \in \Pi$.

<u>Fact</u>: There exists a unique "highest" root, with height $h - 1$.

<u>Theorem</u> (Kostant): The rank numbers are dual to the exponents.

Eg B_3



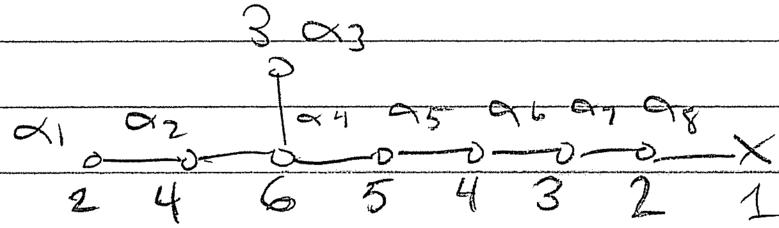
Explicitly: Let λ_k = # roots of height k.

Then $\#\{k : \lambda_k \geq i\} = d_{n-i+1} - 1$

Theorem: Express the highest root ρ in Π coordinates.

The coefficients are the entries of the affine PF eigenvector

Eg E_8



$$\rho = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$$

In particular the entries of the affine PF vector sum to h

Another interpretation of the affine PF vector:

vertices = facet normals to a Euclidean simplex (the fundamental domain of an affine Weyl group).

PF entries = the $(n-1)$-dim volumes of the facets (via the Minkowski condition).

Recall:

PF entries = dimensions of irreps of a finite subgroup of $SU(2)$.

Connection?

THE END.