Tues Apr 23

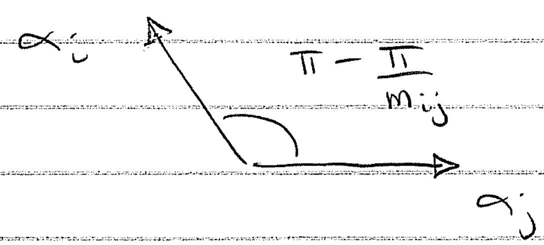
Abstract Reflection Groups

We would like a purely algebraic description of finite reflection groups.
This is due to Coxeter.

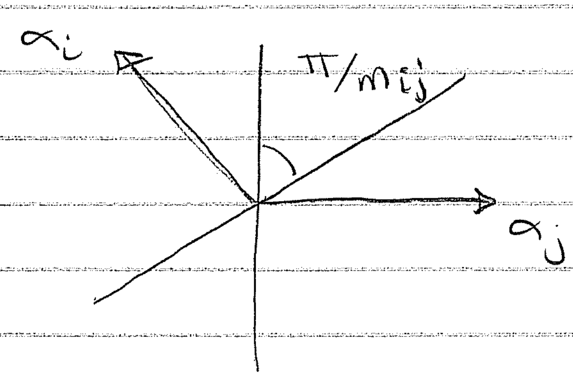
Let G be a FGGR with root data.

$\Pi = \{\alpha_1, \dots, \alpha_n\} \subseteq \Phi^+ \subseteq \Phi$

and simple reflections $s_i := t_{\alpha_i}$.
Define the Coxeter diagram labels m_{ij} by



Equivalently,



Note that the product of reflections $s_i s_j$ is a rotation by angle $2\pi/m_{ij}$, hence $s_i s_j \in G$ has order m_{ij}.

Theorem (Coxeter, 1935):

These are the only relations in the group. That is, we have a "Coxeter" presentation

$$G = \langle s_i : (s_i s_j)^{m_{ij}} = 1 \ \forall i,j \rangle$$

[Note: $m_{ii} = 1$ means $s_i^2 = 1$

$m_{ij} = 2$ means $s_i s_j s_i s_j = 1$

$\iff s_i s_j = s_j s_i$

$m_{ij} = 3$ means $s_i s_j s_i s_j s_i s_j = 1$

$\iff s_i s_j s_i = s_j s_i s_j$

etc.]

Proof: We know that G is generated by simple reflections $s_1, s_2, \ldots, s_n$ and we know that the relations

$$(s_i s_j)^{m_{ij}} = 1$$

are true.

We must show that every relation in the group is a consequence of these. So assume the following equation holds:

$$s_{i_1} s_{i_2} \cdots s_{i_k} = 1 \qquad (*)$$

We will show that (*) follows from local "braid moves" of the form

$$\underbrace{s_i s_j s_i \cdots}_{m_{ij}} \mapsto \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$$

First an example: Consider the symmetric group of order 24.

$$\mathfrak{S}_4 = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_{13})^2 = (s_2 s_3)^3 = 1 \rangle$$

Verify that

$$s_1 s_2 s_3 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3 s_2 = 1$$

using only braid moves.

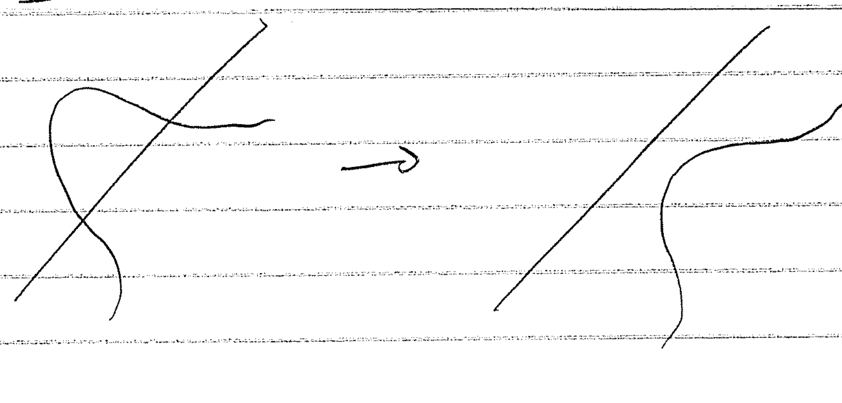
we have

$$s_1 s_2 s_3 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3 s_2$$
$$s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1 s_2 s_3 s_2$$
$$s_1 s_2 s_3 s_1 s_2 s_3 s_2 s_1 \cancel{s_2 s_2} s_3 s_2$$
$$s_1 s_2 s_3 s_1 s_3 s_2 s_3 s_1 s_3 s_2$$
$$s_1 s_2 \cancel{s_3 s_3} s_1 s_2 s_1 \cancel{s_3 s_3} s_2$$
$$s_1 s_2 s_1 s_2 s_1 s_2$$
$$s_2 s_1 \cancel{s_2 s_2} s_1 s_2$$
$$= 1 \qquad \checkmark$$

We must show this always works.
In the general case (*) is a closed topological loop. Since $\mathbb{R}^n$ is simply connected we can shrink the loop to a point.

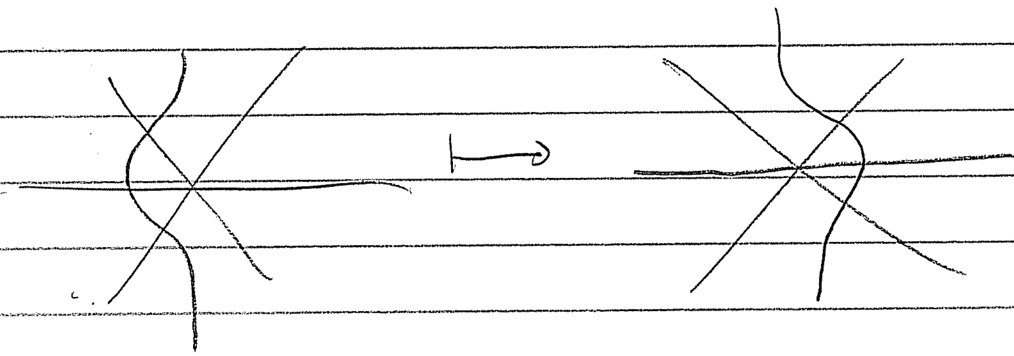
As we shrink we can avoid anything of dim $\leq n-3$. There are 2 cases:

dim $n-1$:



Algebraically, this is $s_i s_i \mapsto 1$

dim n-2 : If our loop crosses an $(n-2)$-dim intersection of hyperplanes, it looks like



Algebraically, this is a braid move.

[Remark: The Coxeter presentation makes FGGR extremely <u>combinatorial</u>. (See Björner - Brenti).]

This theorem suggests the following:

Definition: Given any simple labeled graph with vertices $\{1, 2, \ldots, n\}$ and edge labels $m_{ij} \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ satisfying

- $m_{ij} = m_{ji} \quad \forall i, j$ (undirected)
- $m_{ij} = 1 \iff i = j$ (no loops).

we define the "abstract reflection group"

$$G := \langle s_1, s_2, \ldots, s_n : (s_i s_j)^{m_{ij}} = 1 \ \forall i, j \rangle$$

The standard name for this is a "Coxeter group".

Remarks:

- Since $m_{ii} = 1$ we have $s_i^2 = 1 \ \forall i$, so G is generated by involutions
- The relation $(s_i s_j)^{\infty} = 1$ is shorthand for "no relation".

Good Question:
Is there anything geometric about an abstract Coxeter group?

Answer: Kind of.

Define an abstract Real vector space

$$V := \mathbb{R}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

just symbols

For all α_i, α_j we define

$$B(\alpha_i, \alpha_j) := -2\cos\left(\frac{\pi}{m_{ij}}\right)$$

and extend to a symmetric bilinear form $B : V \times V \to \mathbb{R}$. [By convention, set $-2\cos(\pi/\infty) := -2$.]

For all i we have $B(\alpha_i, \alpha_i) = -2\cos(\pi) = 2 \neq 0$. Since α_i is anisotropic, we have

$$V = \mathbb{R}\alpha_i \oplus \alpha_i^{\perp}.$$

Define a "reflection" $\sigma_i : V \to V$ by

$$\sigma_i(u) := u - B(\alpha_i, u)\,\alpha_i.$$

Note that

- σ_i is linear
- $\sigma_i(\alpha_i) = \alpha_i - B(\alpha_i, \alpha_i)\alpha_i$
  $= \alpha_i - 2\alpha_i = -\alpha_i$
- σ_i fixes $\alpha_i^\perp$ pointwise, since

$B(\alpha_i, u) = 0 \Rightarrow \sigma_i(u) = u - B(\alpha_i, u)\alpha_i = u$.

Thus $s_i \in GL(V)$ and has order 2.
Also check that

$$B(\sigma_i u, \sigma_i v) = B(u,v) \quad \forall u, v \in V, \forall \sigma_i.$$

Thus the group $\langle \sigma_1, \ldots, \sigma_n \rangle < GL(V)$ preserves the form B.

We wish to show that there exists a (necessarily unique) group homomorphism

$$G = \langle s_1, \ldots, s_n \rangle \longrightarrow \langle \sigma_1, \sigma_2, \ldots, \sigma_n \rangle < GL(V)$$

sending $s_i \mapsto \sigma_i \quad \forall i$.

It's enough to show that $(\sigma_i \sigma_j)^{m_{ij}} = 1 \in GL(V)$ for all i, j.

Proof: Assume $1 < m_{ij} < \infty$ and consider the 2D subspace

$$V_{ij} = \mathbb{R}\alpha_i \oplus \mathbb{R}\alpha_j \subseteq V.$$

Note that the group $\langle \sigma_i, \sigma_j \rangle$ stabilizes V_{ij}. Furthermore, $\forall\, u = a\alpha_i + b\alpha_j \in V_{ij}$ we have

$$B(u,u) = a^2 B(\alpha_i, \alpha_i) + 2ab\, B(\alpha_i, \alpha_j) + b^2 B(\alpha_j, \alpha_j)$$
$$= 2a^2 - 4ab \cos(\pi/m_{ij}) + 2b^2$$
$$= 2\left[\left(a - b\cos\left(\frac{\pi}{m_{ij}}\right)\right)^2 + b^2 \sin^2\left(\frac{\pi}{m_{ij}}\right) \right] > 0$$

since $\sin\left(\frac{\pi}{m_{ij}}\right) > 0$ for $1 < m_{ij} < \infty$.

Hence B is pos. def. on V_{ij} and we can identify

$$(V_{ij}, B) = (\mathbb{R}^2, \text{dot product}).$$

It follows that $\sigma_i, \sigma_j \in GL(V_{ij})$ are actual reflections with angle π/m_{ij}; hence

$$|\langle \sigma_i \sigma_j \rangle| = m_{ij}, \quad \text{as desired} \quad \blacksquare$$

We have constructed the geometric representation of the Coxeter group G:

$$\sigma : G \longrightarrow O(V, B)$$
$$s_i \longmapsto \sigma_i$$

With more work we could show:

① The geometric representation is faithful, i.e., $\ker \sigma = \{1\} < G$

② The irreducible summands of σ correspond to connected components of the Coxeter graph.

③ G is finite $\Longleftrightarrow$ B is pos. def.

In this case we can identify G as a FGGR $< O(n)$.

Hence we have

$FGGR < O(n) \equiv$ Finite Coxeter group.

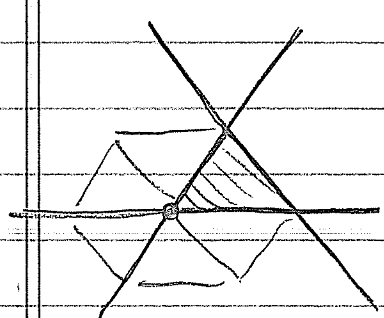
(4) If (V, B) is pos. semidef and for all

$$V_{\langle i \rangle} = \mathbb{R}\{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n\},$$

we have $(V_{\langle i \rangle}, B)$ is pos. def., then G can be realized as a discrete subgroup of $\mathrm{Aff}(n)$ generated by reflections in the walls of a simplex.

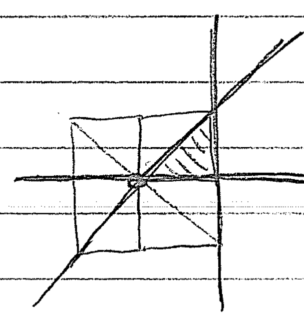
These "affine" Coxeter groups are classified by the "affine" $(p = 2)$ Coxeter diagrams.

Example (rank 2).



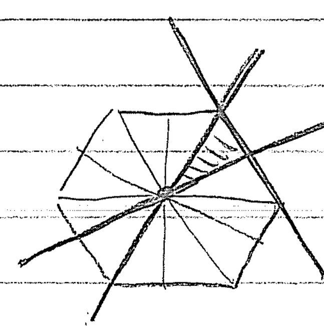
$A_2^{(1)}$

$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$



$B_2^{(1)}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$



$G_2^{(1)}$

$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$

(5) If (V, B) is indefinite, but $(V_{\langle i \rangle}, B)$ is pos. def. for all i,

Then G can be realized as a discrete group of isometries of hyperbolic space, generated by reflections in the walls of a hyperbolic simplex.

These "hyperbolic" Coxeter groups are also classified.

(see handout from Humphries).