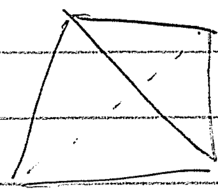
Tues Apr 16

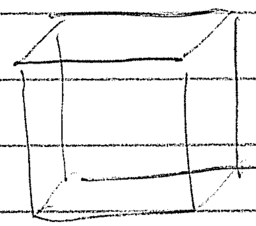
Regular Polytopes

Theorem (before Plato):

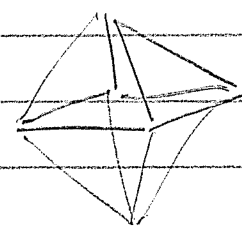
There are exactly 5 regular polyhedra



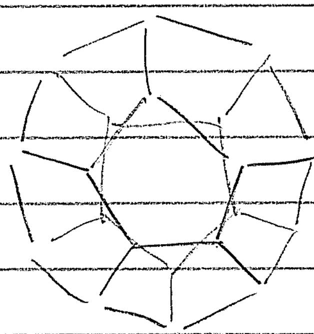
Tetra



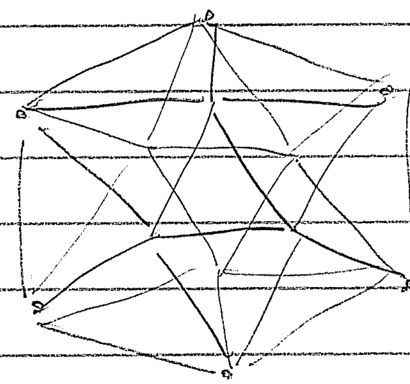
Cube



Octa



Dodeca



Icosa

And That's All!

Problem: Classify regular polytopes in all dimensions.

First: What does "regular" mean?

In 3D: A convex polyhedron is regular if

1. Its faces are congruent regular polygons.
2. The same number of faces meet at each vertex.

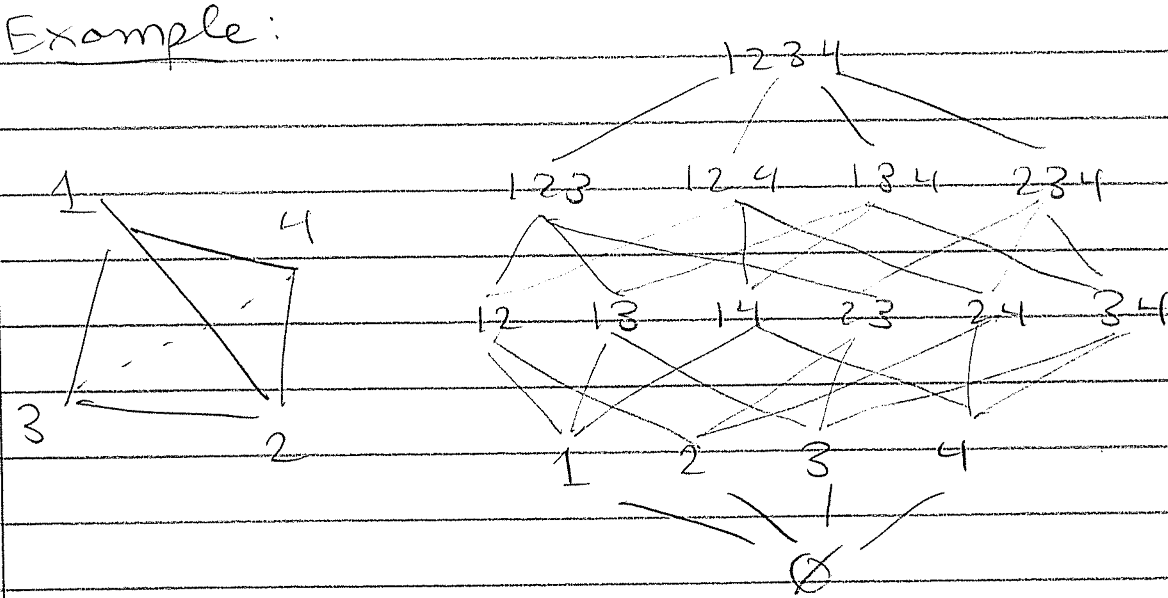
This is a recursive definition.
Today I prefer a non-recursive definition.

Let Q be a convex polytope, decomposed into (open) faces. The faces are partially ordered by inclusion:

Say $F \leq E \iff F \subseteq \overline{E}$.

This is called the "Face lattice" of Q.

Example:



Now define the barycentric subdivison of Q.

$$\Delta(Q) := \{ \text{chains } F_{i_1} < F_{i_2} < \cdots < F_{i_k} \text{ in } Q \}.$$

We can realize $\Delta(Q)$ geometrically as follows: To each face F in Q, associate its barycenter

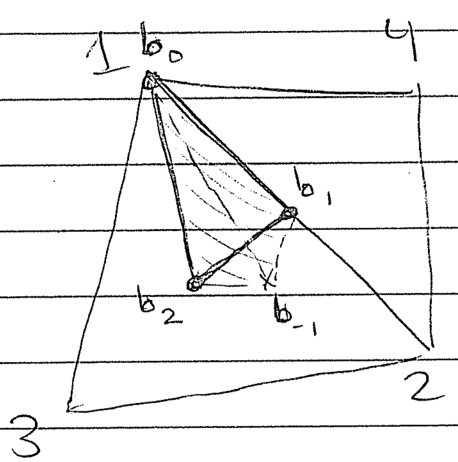
$$b_F = \frac{1}{m}(v_1 + v_2 + \cdots + v_m)$$

where $\{v_1, \ldots, v_m\}$ are the vertices of F. [By convention, let $b_\emptyset = 0$.]

Now to each chain in Q we associate the convex hull of its barycenters

$$F_{i_1} < F_{i_2} < \cdots < F_{i_k} \longleftrightarrow \text{hull}(b_{i_1}, \ldots, b_{i_k}).$$

Example:



$F_2 = 123$.
$F_1 = 12$
$F_0 = 1$
$F_{-1} = \emptyset$

Given a polytope Q, let $G(Q) < O(n)$ be its symmetry group.

<u>Definition</u>: We say that Q is <u>regular</u>, If $G(Q)$ acts <u>transitively</u> on <u>maximal</u> <u>flags</u> (i.e. maximal faces of $\Delta(Q)$).

<u>Remark</u>: This implies that G acts <u>simply</u>-transitively on maximal flags.

<u>Proof</u>: Choose a maximal flag

$$F: \quad \emptyset =: F_{-1} < F_0 < F_1 < \cdots < F_{n-1} < F_n := Q.$$

with barycenters $b_{-1} = 0, b_0 = F_0, b_1, \ldots, b_{n-1}$.
[Without loss we can assume that $b_{-1} = b_n = 0$, i.e. Q is centered at 0.]

Now suppose $gF = F$ for some $g \in G$. We wish to show that $g = 1$. Indeed, since $gF = F$ we have $gF_i = F_i \ \forall i$ and hence $g(b_i) = b_i \ \forall i$.
But $b_{-1} = 0, b_0, b_1, \ldots, b_{n-1}$ are the vertices of a simplex $\Longrightarrow$ $b_0, \ldots, b_{n-1}$ are linearly independent $\Longrightarrow$ $g = 1$

∎

Now given a regular polytope Q, let's examine the group $G := G(Q) < O(n)$.

<u>General Fact</u>: Given faces $E < F$ in Q, the interval

$$[E, F] = \{ D \text{ in } Q : E < D < F \}$$

is itself the face lattice of a polytope of dimension $\dim(F) - \dim(E) - 1$. ///

Thus given a maximal flag

$$F : \emptyset = F_{-1} < F_0 < F_1 < \cdots < F_{n-1} < F_n = Q,$$

the interval $[F_{i-1}, F_{i+1}]$ is the face lattice of a 1D polytope, i.e. a line segment:

$$\begin{tikzcd}[column sep=small]
 & F_{i+1} \arrow[dl, dash] \arrow[dr, dash] & \\
F_i \arrow[dr, dash] & & \text{something else. (call it } F_i'\text{).} \arrow[dl, dash] \\
 & F_{i-1} &
\end{tikzcd}$$

Thus for all $i \in \{0, 1, \ldots, n-1\}$, there exists a unique maximal flag, adjacent in rank i.

$$F_{\langle i \rangle} : \emptyset \leq \cdots < F_{i-1} < F_i' < F_{i+1} < \cdots \leq Q.$$

Since G is simply-transitive on maximal flags, $\exists!\, g \in G$ with $gF = F_{\langle i \rangle}$.

Claim: This g is a reflection.

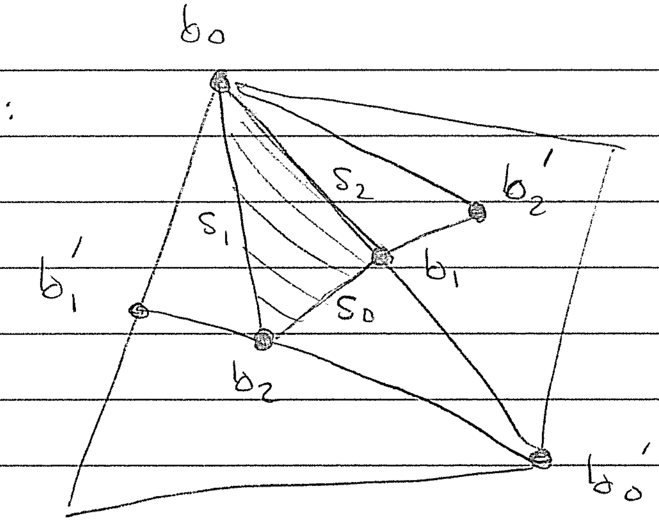
Proof: Since $gF_j = F_j$ for all $j \neq i$, g fixes the barycenters $g(b_j) = b_j$ for $j \neq i$. Hence g fixes the hyperplane

$$\mathbb{R}\langle b_0, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{n-1} \rangle.$$

Since $g \neq 1$ (because $gF_i = F_i' \neq F_i$) and $g \in O(n)$, we conclude that g is a reflection. $\boxtimes$

Now fix a base flag F and let $s_0, s_1, \ldots, s_{n-1}$ be the (unique) reflections such that $s_i F = F_{\langle i \rangle}$

Picture:



Note that $\langle s_0, s_1, \ldots, s_{n-1} \rangle < G$

<u>Claim</u>: In fact, $\langle s_0, \ldots, s_{n-1} \rangle = G$.

<u>Proof</u>: Let $g \in G$ and consider the flag gF. Since Q is connected there exists a "gallery" of adjacent flags:

$$F = F^{(0)} \to F^{(1)} \to \cdots \to F^{(k)} = gF.$$

By induction on k we show that $g \in \langle s_0, \ldots, s_{n-1} \rangle$. [Same proof as before.]

$\blacksquare$

Conclusion: $G = \langle s_0, \ldots, s_{n-1} \rangle$ is a FGGR.

Q: Could it be just *any* FGGR?
A: NO!

If Q is a regular polytope then $G(G)$ is a *special* FGGR.

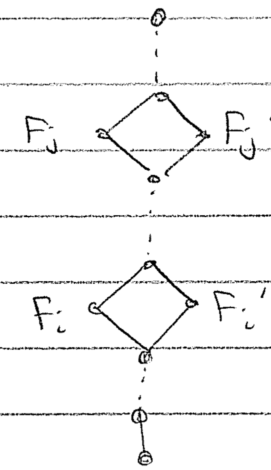
Fix a base flag $F: F_0 < F_1 < \cdots < F_{n-1}$ with barycenters $b_0, b_1, \ldots, b_{n-1}$ and reflections $s_i F = F_{\langle i \rangle}$. Then

Theorem: If $|i-j| \geq 2$ then $s_i s_j = s_j s_i$, and if $|i-j| = 1$ then $s_i s_j \neq s_i s_j$.

Proof: Let $|i-j| \geq 2$ and assume $i+1 < j$. Then s_j preserves each $F_0, F_1, \ldots, F_{i+1}$, hence *fixes* each $b_0, b_1, \ldots, b_{n-1}$, hence fixes $F_{i+1} = \text{hull}(b_0, \ldots, b_{i+1})$ pointwise. Then since $F_i' < F_{i+1}$ we have $s_j F_i' = F_i$

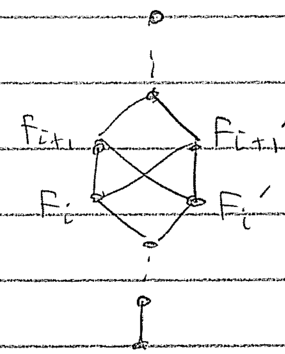
Similarly we have $s_i F_j' = F_j'$. Indeed, s_i fixes the quotient space $\mathbb{R}^n / \mathbb{R}F_{j-1}$ since it preserves F_{j-1} and fixes the basis $b_j, \ldots, b_{n-1}$. Then since $F_{j-1} < F_j'$ we have $s_i F_j' = F_j'$.

Picture:



commute.

If $|i-j|=1$ then we get → which contradicts polytopality.



Conclusion: Since reflections s_i, s_j commute $\Longleftrightarrow$ their normal vectors are $\perp$, we conclude that the Coxeter diagram of G is a chain (has no branches).

So we have, ...

Theorem (Schläfli, 1855):
The regular polytopes correspond to the following Coxeter diagrams.

A_n	= o—o—o - - - o—o	n-simplex
B_n	= o—o—o - - - o—o$\overset{4}{—}$o	n-cube / n-octahedron.
F_4	= o—o$\overset{4}{—}$o—o	24-cell
H_3	= o$\overset{5}{—}$o—o	dodec / icos.
H_4	= o$\overset{5}{—}$o—o—o	600-cell / 120-cell
$I_2(m)$	= o$\overset{m}{—}$o	regular m-gon

And That's All!

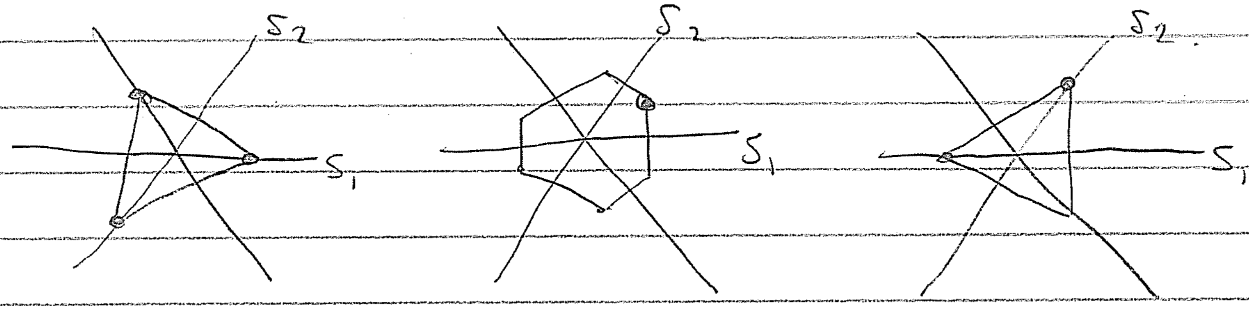
Proof: We showed these are the only possibilities. Now we must construct them. (This goes back to Euclid.)

The nice proof uses Wythoff's Construction.

- Consider any Coxeter diagram
- Choose a subset of vertices/hyperplanes.
- Consider a point in $\overline{C}$ touching exactly these walls.
- Consider the convex hull of the orbit of this point.

The result is some polytope.

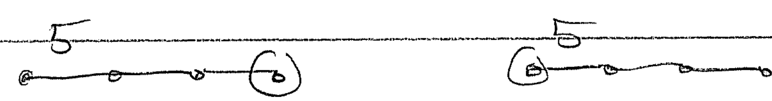
Example : Type A_2.



$s_1 - s_2$ $\qquad$ $s_1 - s_2$ $\qquad$ $s_1 - s_2$

Fact : The regular polytopes are obtained by choosing one endpoint of a chain diagram. The two choices give dual polytopes.

Example :



600 - cell $\qquad$ 120 - cell

More to say later . . .