Thurs Apr 11

BIG DAY, Part 2

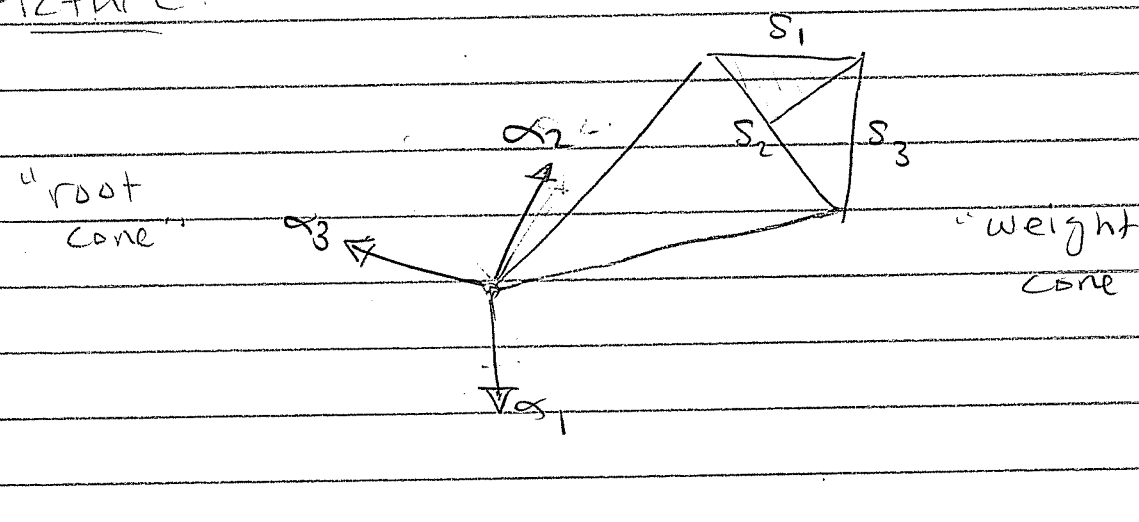
From here on out I will not provide full proofs (because we don't have time). Hopefully I have adequately prepared you for this moment...

Consider a triple (G, Σ, Φ) of FGGR, FMS, FRS. Then G is generated by the simple reflections

$$s_i := t_{\alpha_i}, \quad \text{where}$$

$\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \Phi^+ \subseteq \Phi$ are the simple roots $\perp$ to the walls of the fundamental chamber (which is a simplicial cone).

Picture:



Goal: Classify the possible cones that can occur.

We will encode the cone via its <u>Gram matrix</u> $A^t A$, where $A = (\alpha_1 \cdots \alpha_n)$.

Since root lengths don't matter (today) we will assume that

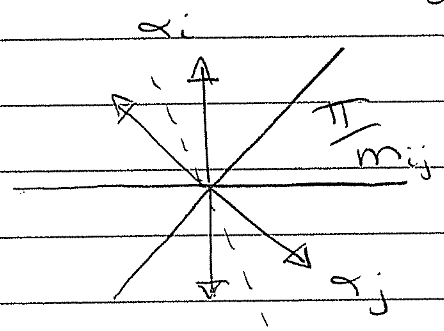
$$\|\alpha_i\|^2 = 2 \quad \forall\, i = 1, 2, \ldots, n.$$

Hence the i,j entry of $A^t A$ is

$$\alpha_i \cdot \alpha_j = \|\alpha_i\| \cdot \|\alpha_j\| \cos \theta_{ij}$$
$$= 2 \cos \theta_{ij}$$

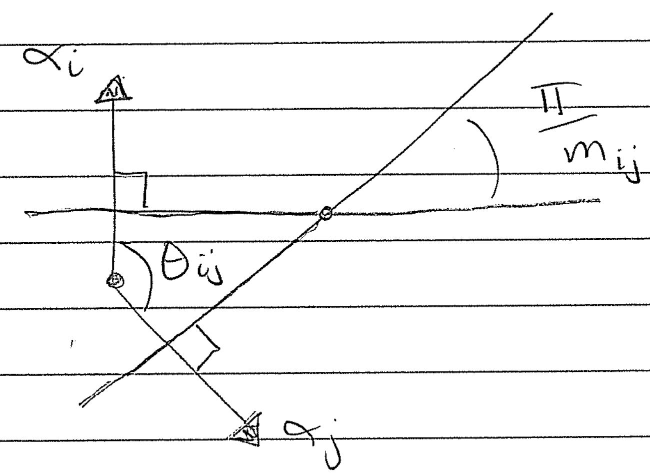
What could the angle $\theta_{ij} = \text{angle}(\alpha_i, \alpha_j)$ be?

Note: The roots α_i, α_j for $i \neq j$ generate a rank 2 (dihedral) sub-root system.



for some $m_{ij} \geq 2$

Better Picture:



Hence we have

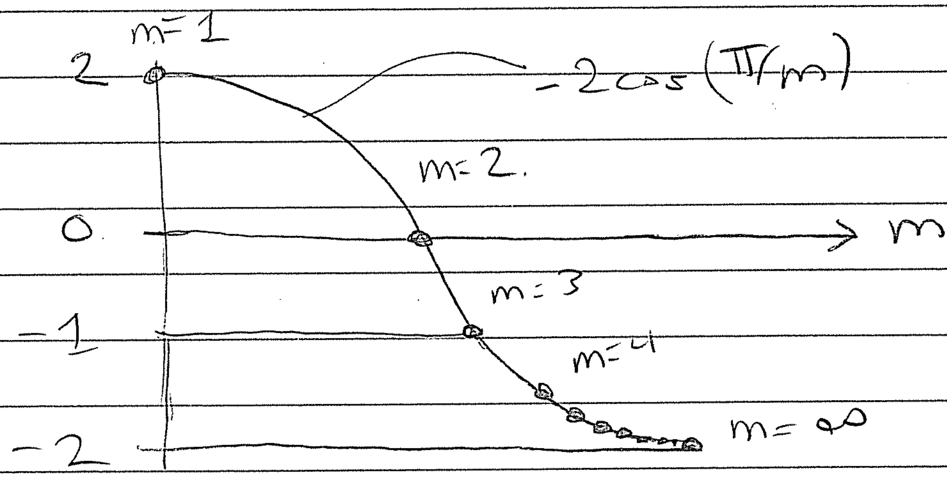
$$\theta_{ij} = \pi - \frac{\pi}{m_{ij}}, \quad \text{and.}$$

$$2\cos\theta_{ij} = -2\cos\left(\frac{\pi}{m_{ij}}\right)$$

Since $\alpha_i \circ \alpha_i = \|\alpha_i\|^2 = 2 = -2\cos\left(\frac{\pi}{1}\right)$
we can safely define $m_{ii} = 1 \ \forall i$.

Then $A^t A = \left(-2\cos\left(\frac{\pi}{m_{ij}}\right)\right)_{i,j=1,\dots,n}$

<u>Observe</u>



[Important Remark: $-2\cos\left(\frac{\pi}{m}\right)$ is negative and strictly decreasing for $m = 2, 3, 4, 5, \ldots$]

<u>Example</u>: My Favorite Cone.

$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \ldots,\ \alpha_n = e_n - e_{n+1}$.

Gram matrix $A^t A$

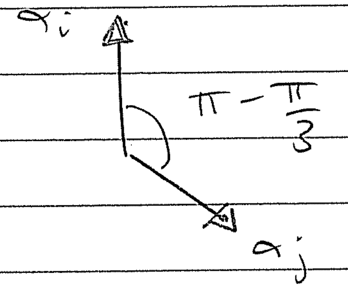
$$= \begin{pmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

Note that $\|\alpha_i\|^2 = 2\ \forall\, i = 1, \ldots, n$.

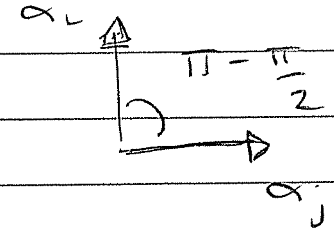
The i,j entry of $A^t A$ is

$$\alpha_i \cdot \alpha_j = \begin{cases} 2 = -2\cos\left(\frac{\pi}{1}\right) & i = j \\ -1 = -2\cos\left(\frac{\pi}{3}\right) & |i-j| = 1 \\ 0 = -2\cos\left(\frac{\pi}{2}\right) & |i-j| > 1. \end{cases}$$

The angles are.



$|i-j| = 1$



$|i-j| > 1$.

See the Zometools.

Important: This is related to the adjacency matrix of the n-path.

$$C = -A^tA + 2I = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}$$

Hence, if $\lambda_1, \dots, \lambda_n$ are the e.values of A^tA then $2-\lambda_1, \dots, 2-\lambda_n$ are the e.values of C.

Correspondence:

E.values of C are < 2 $\Longleftrightarrow$ E.values of $A^t A$ are > 0.

This is certainly true ~~because $\alpha_1, \alpha_2, \ldots, \alpha_n$~~ lin ind. ~~$\Longrightarrow A^t A$ is pos. def.~~

This is Coxeter's Big Idea.

Given any (G, Σ, Φ) with $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ consider the Gram matrix

$$A^t A = \left(-2\cos\left(\frac{\pi}{m_{ij}}\right)\right)_{i,j=1,\ldots,n}$$

and define the Coxeter graph with

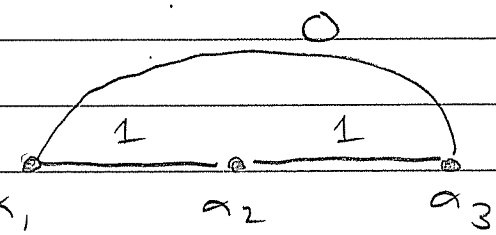
vertices $\longleftrightarrow$ Π

and edges weighted by the numbers

$$\underset{\alpha_i}{\circ}\xrightarrow{-2\cos\left(\frac{\pi}{m_{ij}}\right)}\underset{\alpha_j}{\circ}$$

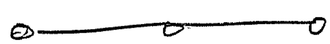
where $\text{angle}(\alpha_i, \alpha_j) = \pi - \frac{\pi}{m_{ij}}$

Example: $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$.



I lied. Since m_{ij} is simpler to write than $-2\cos(\pi/m_{ij})$, the Coxeter graph is drawn with edge labels m_{ij} and the conventions:

$m_{ii} = 1$ but we don't draw this
$m_{ij} = 2$ (weight 0) $\rightarrow$ no edge
$m_{ij} = 3$ (weight 1) $\rightarrow$ unlabeled edge.

So we have 

A labeled graph with these conventions is called a Coxeter Diagram.

We remember the correspondence

edge label = edge weight

m_{ij} $\qquad$ $-2\cos\left(\frac{\pi}{m_{ij}}\right)$

Fact: The (weighted) adjacency matrix of the Coxeter graph is just

$$C = -A^t A + 2I.$$

Call this the "Coxeter adjacency matrix"

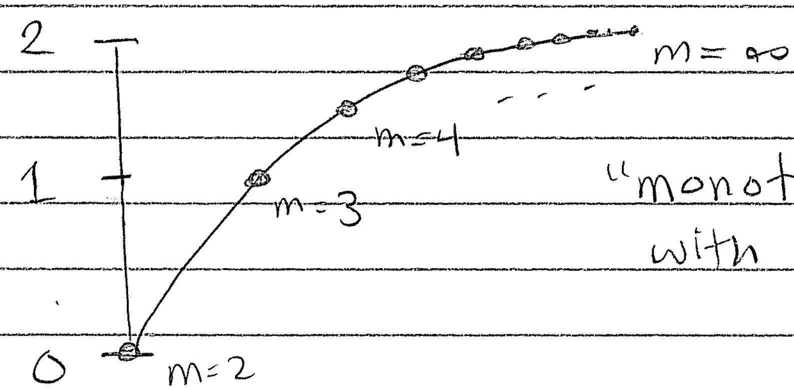
Theorem: The spectral radius satisfies

$$\rho(C) < 2$$

Proof: Since $\alpha_1, \ldots, \alpha_n$ are linearly ind., the Gram matrix $A^t A$ is pos. definite $\blacksquare$

Goal: Classify edge-weighted graphs with weights in the set

$$\left\{ \begin{array}{cccc} 2\cos\left(\frac{\pi}{2}\right), & 2\cos\left(\frac{\pi}{3}\right), & 2\cos\left(\frac{\pi}{4}\right), & \ldots \\ 0, & 1, & \sqrt{2}, & \ldots \end{array} \right\}$$



"monotone with m"

which satisfy $\rho(C) < 2$.

Our tool is Perron-Frobenius.

Recall (PF): If A, B are square matrices with $0 \leq A \leq B$ componentwise and $A \neq B$. Then $\rho(A) \lneq \rho(B)$.

<u>Corollary</u>: If $C \neq D$ are Coxeter diagrams with $m_{ij}^C \leq m_{ij}^D$ $\forall\ i,j$, then

$$\rho(C) \lneq \rho(D).$$

<u>Proof</u>: The entries of the Cox. adjacency matrix are $\begin{cases} 0 & i=j \\ 2\cos\left(\frac{\pi}{m_{ij}}\right) & i \neq j \end{cases}$

and $2\cos\left(\frac{\pi}{m}\right)$ is monotone strictly increasing as a function of m

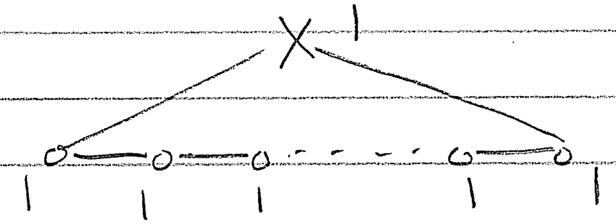
$\blacksquare$

Our Strategy: Find the Coxeter graphs / diagrams C with

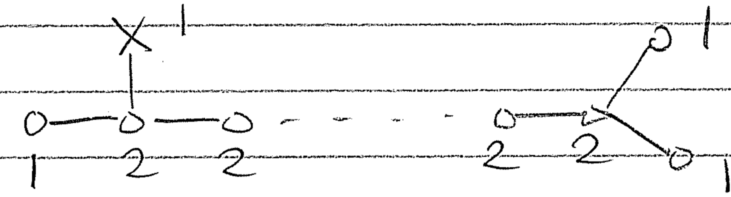
$$\rho(C) = 2$$

Here they are, with the PF vector shown:

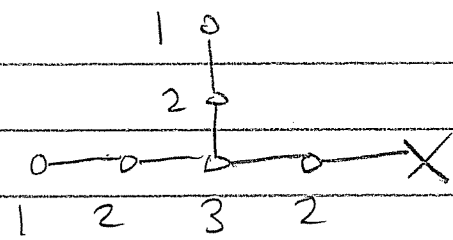
$A_n^{(1)}$ =



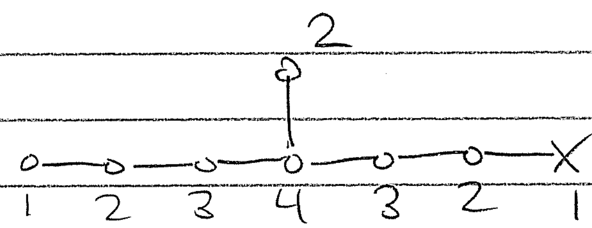
$D_n^{(1)}$ =



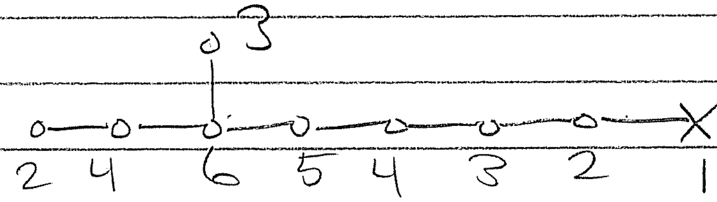
$E_6^{(1)}$ =



$E_7^{(1)}$ =



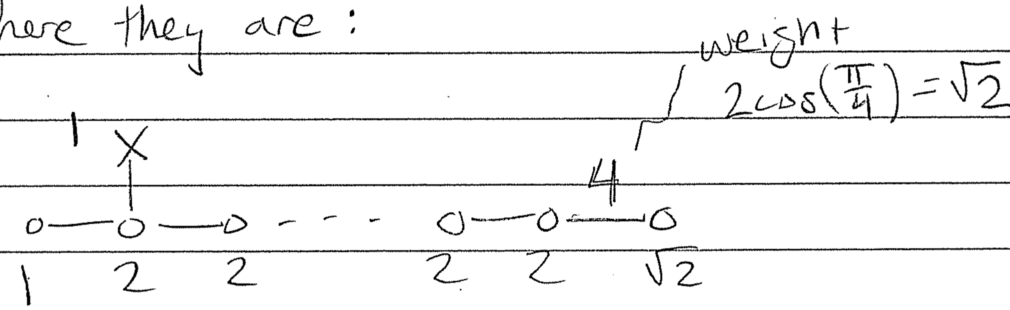
$E_8^{(1)}$ =



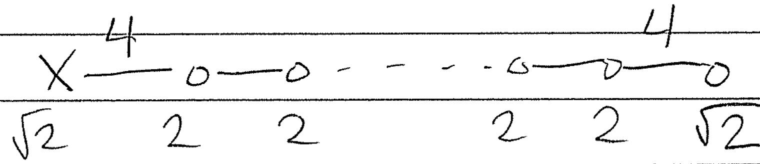
We already knew these.
Are there more?

Yes, and here they are:

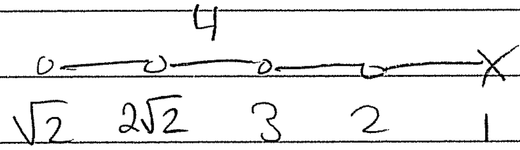
$B_n^{(1)} =$



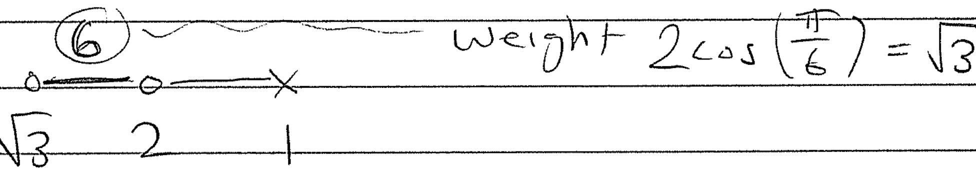
$C_n^{(1)} =$



$F_4^{(1)} =$



$G_2^{(1)} =$



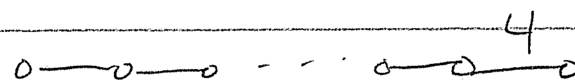
Theorem: That's All!

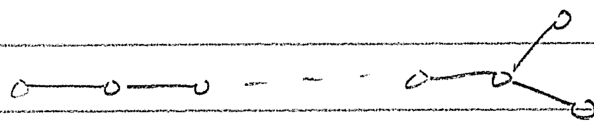
Proof: You know how it goes. ∎

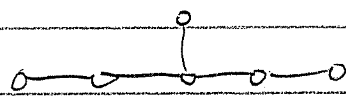
To obtain the diagrams with $\rho(C) < 2$, lower the edge labels in any of the above $\rho(2) = 2$ diagrams.

Theorem (Coxeter, 1935):
The FGGRs are exactly

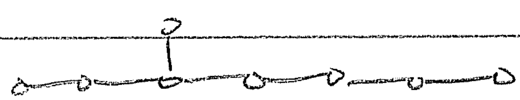
A_n = 

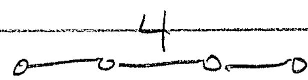
$B_n (= C_n)$ = 

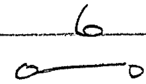
D_n = 

E_6 = 

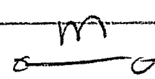
E_7 = 

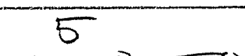
E_8 = 

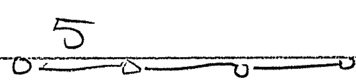
F_4 = 

G_2 =  (special)

Together with the slightly tricker cases:

$G_2(m)$ =  for $m=5$, $m \geq 7$

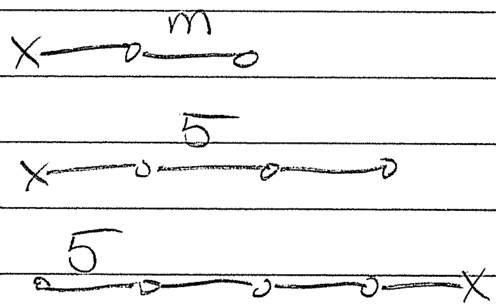
H_3 = 

H_4 = 

And That's All!

Proof:

① Show that these are the only possibilities by examining the $\rho(C)=2$ ("affine") diagrams, and these bad guys:



which have exactly one e.value > 2 and the rest < 2 (they are called "hyperbolic").

② Show that the resulting types actually exist by constructing them (somehow).

See: Euclid's Book XIII for the rank 3 cases.

∎