Tues Apr 9

Recall: A (finite) root system is a finite set of vectors Φ such that

① $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ $\quad \forall\, \alpha \in \Phi$

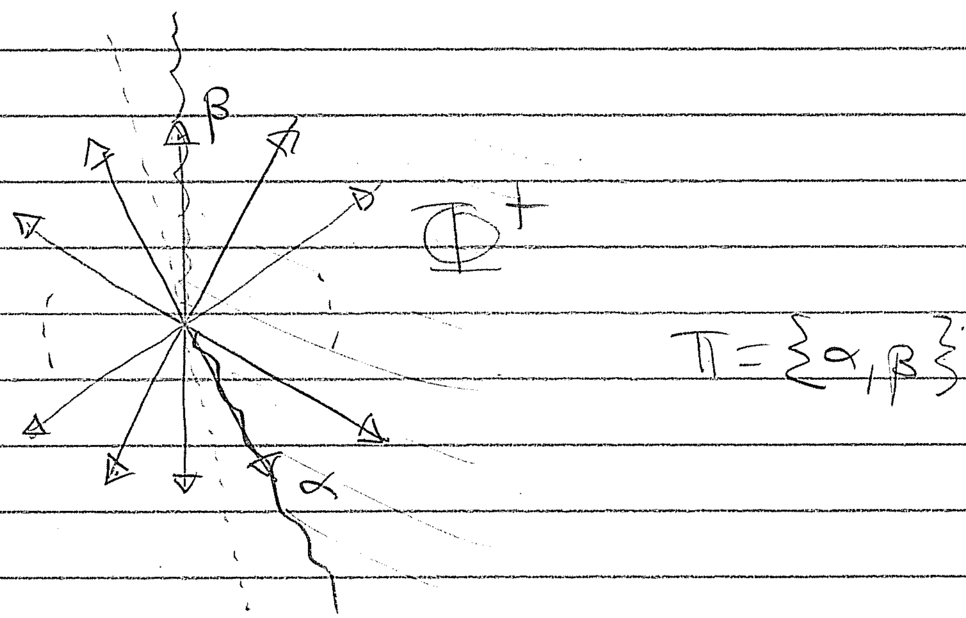
② $t_\alpha(\beta) \in \Phi$ $\quad \forall\, \alpha, \beta \in \Phi$

For each generic choice of direction we define positive and simple roots

$$\Phi \supseteq \Phi^+ \supseteq \Pi.$$

The rank of Φ is $|\Pi| = \dim(\mathbb{R}\Phi)$.

The rank 2 root systems look like.



with 2m equiangular pairs of unit vectors.

[Remark: This is the root system of type $G_2(m)$]

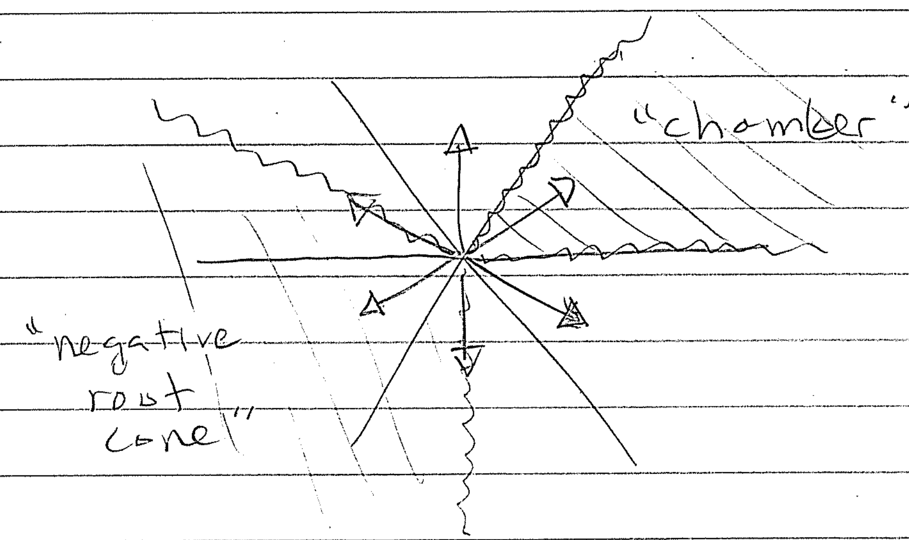
The root system determines a reflection group

$$G(\Phi) = \langle t_\alpha : \alpha \in \Phi \rangle$$
$$= \langle t_\alpha : \alpha \in \Phi^+ \rangle$$
$$\left(= \langle t_\alpha : \alpha \in \Pi \rangle \right) \text{ I.O.U.}$$

and a system of mirrors

$$\Sigma(\Phi) = \{ \alpha^\perp : \pm\alpha \in \Phi \}.$$

The root system an mirror system are "dual" in a nice way:



It looks like:

- Each (polyhedral) chamber is dual to a (finitely generated) negative root cone.
- The chambers are all equivalent

Let's prove this.

Definition: The chambers of Σ are the connected components of

$$V \setminus \bigcup_{H \in \Sigma} H.$$

Given $\alpha \in \Phi$ we define the positive and negative half-spaces

$$V_\alpha^+ := \{ x \in V : \alpha \cdot x > 0 \}$$
$$V_\alpha^- := \{ x \in V : \alpha \cdot x < 0 \}$$

Clearly each chamber C is a polyhedral cone: Choose any $x \in C$ and define the corresponding positive roots

$$\Phi^+ := \{ \alpha \in \Phi : x \cdot \alpha > 0 \}.$$

Then we have

$$C = \bigcap_{\alpha \in \Phi^+} V_\alpha^+ ,$$

which is polyhedral.

<u>Theorem</u>: In fact we have

$$C = \bigcap_{\alpha \in \Pi} V_\alpha^+$$

where $\Pi \subseteq \Phi^+$ is the unique simple system.

<u>Proof</u>: Let $C' = \bigcap_{\alpha \in \Pi} V_\alpha^+$.

Clearly we have $C \subseteq C'$.
Suppose for contradiction that $C \subsetneq C'$.

Then one of the bounding hyperplanes H_ρ of C (for some $\rho \in \Phi$) intersects C' nontrivially. That is, there exists some $x \in C'$ such that $x \cdot \rho = 0$. (i.e. $x \in H_\rho$).

But recall that

$$\rho = \sum_{\alpha \in \Pi} a_\alpha \alpha ,$$

where $a_\alpha > 0 \ \forall \alpha \in \Pi$
or $a_\alpha < 0 \ \forall \alpha \in \Pi$.

Furthermore, since $x \in C'$ we have
$x \cdot \alpha > 0 \ \forall \alpha \in \Pi$.

Hence

$$x \cdot \rho = \sum_{\alpha \in \Pi} a_\alpha (x \cdot \alpha) \neq 0 .$$

Contradiction ∎

Thus each chamber has the form

$$C = \bigcap_{\alpha \in \Pi} V_\alpha^+$$

for a simple system Π, and the <u>walls</u> of C are supported on the hyperplanes H_α, for $\alpha \in \Pi$.

Corollary: In fact, we have a bijection

choices of simple system $\longleftrightarrow$ chambers of Σ.

Proof: Given a chamber C we saw that

$$C = \bigcap_{\alpha \in \Pi} V_\alpha^+$$

for some simple system. Conversely, choose any simple system Π. Since Π is lin. ind. it is contained in an open half-space, i.e. $\exists\, x$ such that $x \cdot \alpha > 0 \quad \forall\, \alpha \in \Pi$.

But then $x \cdot \rho > 0$ for all $\rho \in \Phi^+$. Hence x is contained in some chamber C, and this chamber satisfies

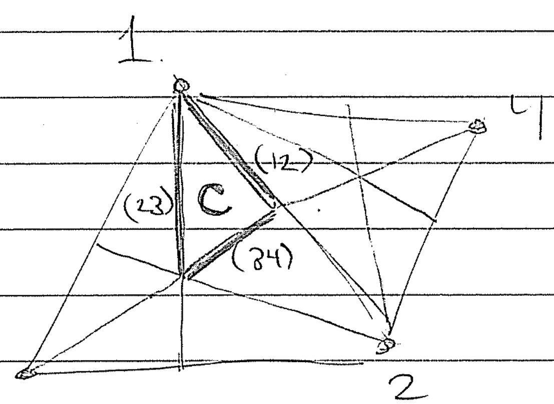
$$C = \bigcap_{\alpha \in \Pi} V_\alpha^+$$

$\blacksquare$

Now let (C, Π) be any choice of chamber / simple roots with reflection group G.

Theorem: The group G is generated by the "simple reflections" t_α, $\alpha \in \Pi$, in the walls of the "fundamental chamber" C.

Example



$\mathfrak{S}_4$ is generated by the adjacent transpositions $(12), (23), (34)$.

Proof of Theorem: Let $\{\alpha_1, \ldots, \alpha_n\} = \Pi$ and consider the simple reflections $s_i := t_{\alpha_i}$. Recall that

$$G = \langle t_\alpha : \alpha \in \Phi \rangle.$$

Define the subgroup.

$$G' := \langle s_1, \ldots, s_n \rangle < G.$$

We will show that $G' = G$.
First we will show that every chamber has the form gC for some $g \in G'$.

Say two chambers are adjacent if they share a wall. Now let D be any chamber. Since space is connected, $\exists$ a "gallery"

$$C = C_0, C_1, C_2, \ldots, C_\ell = D$$

with C_i, C_{i+1} adjacent $\forall i$.
Since C_1 is adjacent to C, we have $C_1 = s_j C$ for some simple s_j.
Now assume $C_i = gC$ for $g \in G'$.
The walls of gC correspond to reflections $gs_1g^{-1}, \ldots, gs_ng^{-1}$ because

$$\boxed{g(H_t) = H_{gtg^{-1}}}, \text{ as you know.}$$

Since C_{i+1} is adjacent to $C_i = gC$, we have

$$\begin{aligned} C_{i+1} &= g s_j g^{-1} C_i \qquad \text{for some } s_j \\ &= g s_j \cancel{g^{-1} g}\, C \\ &= g s_j C \end{aligned}$$

with $g s_j \in G'$. By induction we conclude that $D = gC$ for some $g \in G'$.

Next we will show that $G' = G$

Let t_ρ be any reflection, $\rho \in \Phi$. The wall H_ρ bounds some chamber D. But we know that $D = gC$ for some $g \in G'$ and the walls of D are the reflections $g s_1 g^{-1}, \ldots, g s_n g^{-1}$.

We conclude that $t_\rho = g s_j g^{-1} \in G'$ for some j. Hence

$$G = \langle t_\rho : \rho \in \Phi \rangle < G',$$

as desired. $\blacksquare$

Corollary 1: G acts transitively on the chambers.

Corollary 2: G acts transitively on simple systems

Corollary 3: If t is any reflection in G, then t is conjugate to a simple reflection.

Finally, we will show that G acts simply, transitively on chambers

Theorem: Given any two chambers C, D there exists a unique $g \in G$ such that $D = gC$.

Proof: WLOG assume C is the fundamental chamber. If

$$D = g_1 C = g_2 C$$

Then $g_1^{-1} g_2 C = C$. So we are done if we can show that $gC = C \Rightarrow g = 1$.

Let $g = s_1 s_2 \cdots s_k$ be a reduced word for g (i.e. k minimal) in the simple reflections. This determines a closed path of adjacent chambers

$$C \to s_1 C \to s_1 s_2 C \to \cdots \to s_1 s_2 \cdots s_k C = C.$$

At each step we cross a hyperplane corresponding to a (possibly non-simple) reflection

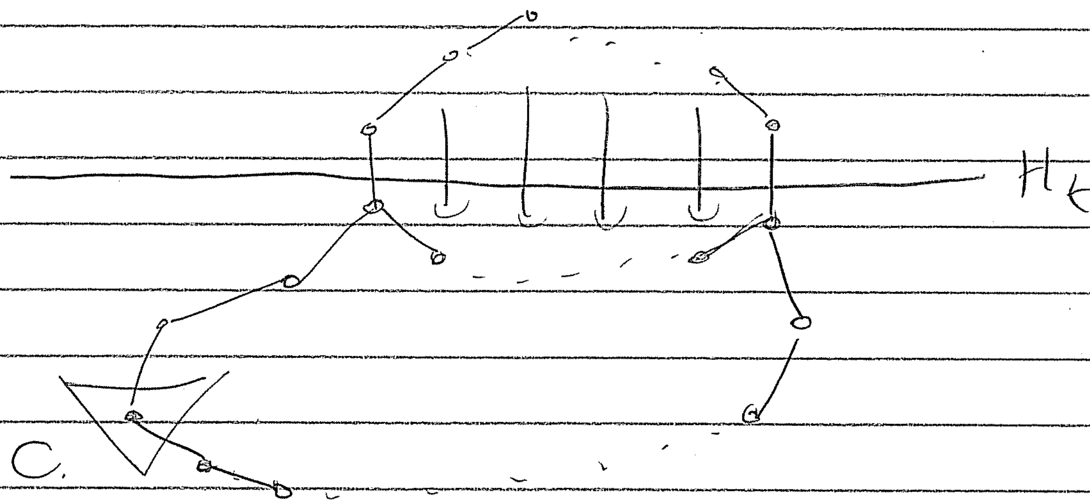
$$C \to t_1 C \to t_2 t_1 C \to \cdots \to t_k \cdots t_2 t_1 C = C.$$

[The fact that $t_i \cdots t_2 t_1 = s_1 s_2 \cdots s_i \;\; \forall i$ means that

$$t_i = s_1 s_2 \cdots s_{i-1} s_i s_{i-1} \cdots s_1$$
(If you want to know...)]

Since the walk is closed, it eventually crosses each hyperplane an even number of times.

Consider any such repetition:



This gives a word

$$g = \underbrace{t_k \cdots t_{j+1}\, t\, t_{j-1} \cdots t_{i+1}\, t}\, t_{i-1} \cdots t_2 t_1$$

Reflect this segment across H_t to get a path that is 2 steps shorter

$$g = \underbrace{t_k \cdots t_{j+1} (t t_{j-1} t) \cdots (t t_{i+1} t) t_{i-1} \cdots t_2 t_1}_{\text{Now there are } k-2 \text{ steps}}$$

Continue until we reach $g = 1$

$\blacksquare$