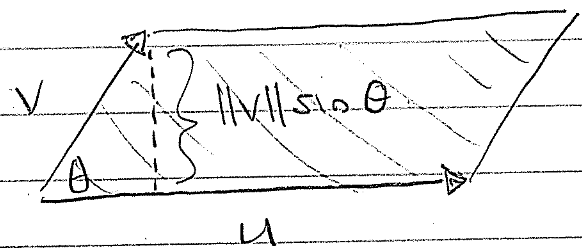
Tues Apr 2

Gram Matrices and Volume

What is the area of a parallelogram?



$$\text{Area} = \text{base} \cdot \text{height}$$
$$= \|u\| \|v\| \sin\theta$$

$$\text{Area}^2 = \|u\|^2 \|v\|^2 \sin^2\theta$$
$$= \|u\|^2 \|v\|^2 (1 - \cos^2\theta)$$
$$= \|u\|^2 \|v\|^2 - \|u\|^2 \|v\|^2 \cos^2\theta$$
$$= (u \cdot u)(v \cdot v) - (u \cdot v)^2$$

$$= \det \begin{pmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{pmatrix}$$

$$= \det(A^t A)$$

where $A = \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$.

Theorem: This works in general. That is, given lin. ind. vectors $\alpha_1, \dots, \alpha_m \in \mathbb{R}^n$, the m-dimensional "volume" of the "parallelogram"

$$\langle \alpha_1, \dots, \alpha_m \rangle = \{ t_1\alpha_1 + \dots + t_m\alpha_m \;;\; 0 \le t_i \le 1 \;\; \forall i \}$$

equals $\sqrt{\det(A^tA)}$, where $A = n\Big\{ \underbrace{\begin{pmatrix} \alpha_1 & \cdots & \alpha_m \end{pmatrix}}_{m}$

Proof by induction on m:

For $m=1$ we have $\mathrm{Vol}\langle \alpha_1 \rangle = \sqrt{\det(\alpha_1^t \alpha_1)}$
$= \sqrt{\det(\|\alpha_1\|^2)} = \sqrt{\|\alpha_1\|^2} = \|\alpha_1\|$ ✓.

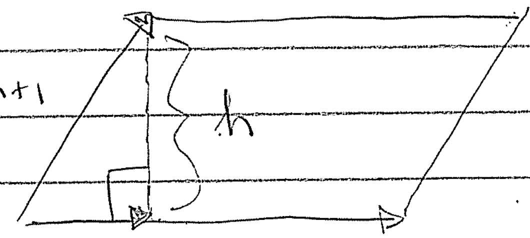
Now consider lin. ind. $\alpha_1, \dots, \alpha_{m+1} \in \mathbb{R}^n$ and let

$$A = (\alpha_1 \cdots \alpha_m).$$

The volume of $\langle \alpha_1, \dots, \alpha_{m+1} \rangle$ is

Vol(Base) · Height

Picture.



The projection of α_m onto $\langle \alpha_1, \ldots, \alpha_m \rangle$ is Ax for <u>some</u> $x \in \mathbb{R}^m$ such that

$$A^t(\alpha_{m+1} - Ax) = 0$$

$$A^t \alpha_{m+1} = A^t A x$$

Note that

$$\begin{aligned} h^2 &= \|\alpha_{m+1} - Ax\|^2 \\ &= (\alpha_{m+1} - Ax)^t (\alpha_{m+1} - Ax) \\ &= (\alpha_{m+1} - Ax)^t \alpha_{m+1} - \underbrace{\cancel{(\alpha_{m+1} - Ax)^t}}_{0} Ax \\ &= \alpha_{m+1}{}^t \alpha_{m+1} - x^t A^t \alpha_{m+1} \\ &= \alpha_{m+1}{}^t \alpha_{m+1} - x^t A^t A x. \end{aligned}$$

Thus if we define $A_+ = (\alpha_1 \cdots \alpha_m \, \alpha_{m+1})$, then

$$A_+^t A_+ = \left(\begin{array}{c|c} A^t A & A^t \alpha_{m+1} \\ \hline \alpha_{m+1}^t A & \alpha_{m+1}^t \alpha_{m+1} \end{array}\right)$$

$$= \left(\begin{array}{c|c} A^t A & A^t A x \\ \hline x^t A^t A & x^t A^t A x + h^2 \end{array}\right)$$

Finally, $\det(A_+^t A_+) =$

$$\det \left(\begin{array}{c|c} A^t A & A^t A x \\ \hline x^t A^t A & x^t A^t A x \end{array}\right) + \det \left(\begin{array}{c|c} A^t A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline x^t A^t A & h^2 \end{array}\right)$$

column m+1 is combination of the first m columns

$$= 0 + \det(A^t A) \cdot h^2$$

By induction, we have.

$$\mathrm{Vol}\langle \alpha_1, \ldots, \alpha_{m+1} \rangle = \mathrm{Vol}\langle \alpha_1, \ldots, \alpha_m \rangle \cdot h$$

$$= \sqrt{\det(A^t A)} \cdot h$$

$$= \sqrt{\det(A_+^t A_+)}$$

Corollary 1: Given a basis $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^n$, the full-dimensional parallelogram has volume

$$\mathrm{Vol}\langle \alpha_1, \ldots, \alpha_n \rangle = \sqrt{\det(A^t A)}$$

$$= \sqrt{\det(A^t)\det(A)}$$

$$= \sqrt{\det(A)^2}$$

$$= |\det(A)|$$

Corollary 2: Given lin. ind. $\alpha_1, \dots, \alpha_m \in \mathbb{R}^n$, define the simplex

$$\Delta(\alpha_1, \dots, \alpha_m) = \left\{ \sum t_i \alpha_i : 0 \le t_i \le 1 \ \forall i, \ \sum t_i \le 1 \right\}.$$

Then the m-dim "volume" of the simplex is

$$\mathrm{Vol}\, \Delta(\alpha_1, \dots, \alpha_m) = \frac{1}{m!} \sqrt{\det(A^t A)}$$

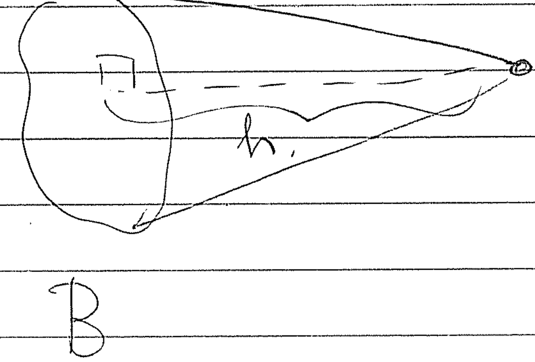
where $A = (\alpha_1 \cdots \alpha_m)$.

Proof: If we can show that

$$\mathrm{Vol}\, \Delta(\alpha_1, \dots, \alpha_m) = \frac{1}{m} \mathrm{Vol}\, \Delta(\alpha_1, \dots, \alpha_{m-1}) \cdot \text{height}$$

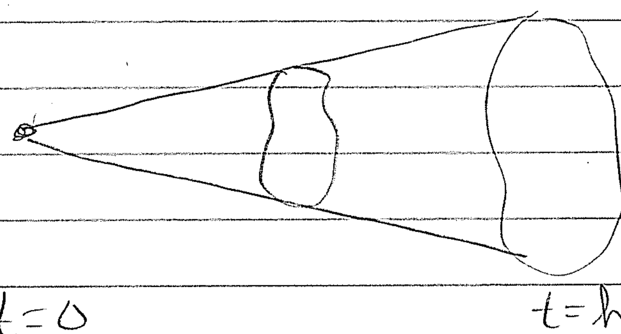
then the result follows by induction.

More generally consider any convex $(m-1)$-dim. solid B. and cone over it.



Let B_+ be the m-dim solid cone.

Consider the orthogonal $(m-1)$-dim slices



The slice at t is dilated by factor t/h in $(m-1)$-dimensions, hence it has volume

$$\mathrm{vol}(B) \cdot \left(\frac{t}{h}\right)^{m-1}$$

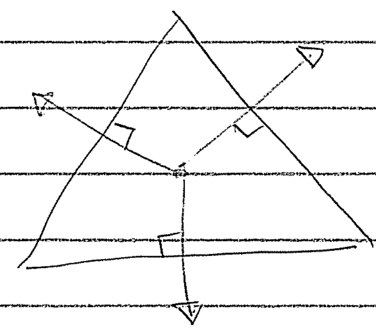
Now integrate:

$$\mathrm{Vol}(B_+) = \int_0^h \mathrm{vol}(B) \left(\frac{t}{h}\right)^{m-1} dt.$$

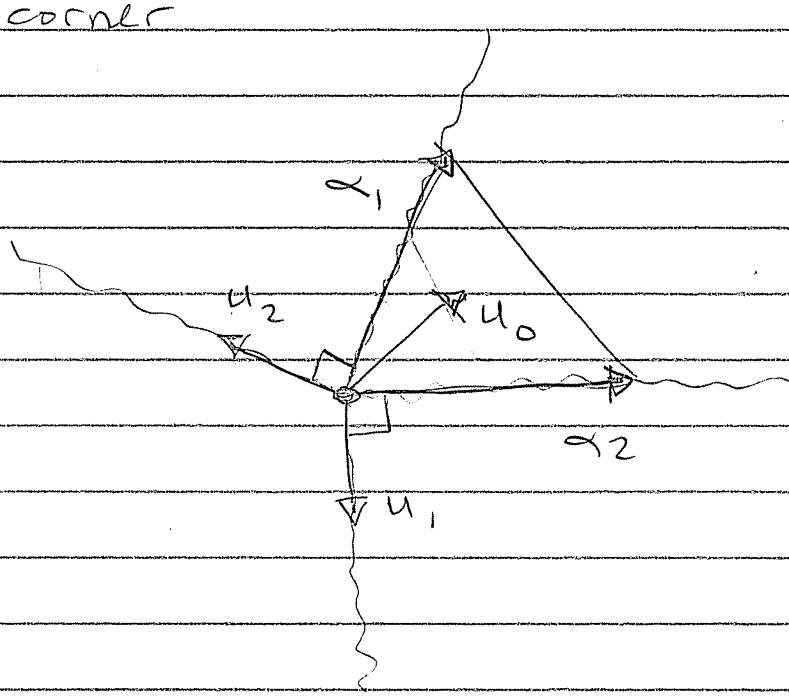
$$= \frac{\mathrm{vol}(B)}{h^{m-1}} \left[\frac{t^m}{m} \right]_0^h = \frac{1}{m} \mathrm{vol}(B) \cdot h.$$

$\blacksquare$

Finally, consider the simplex $\Delta(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{R}^n$ with <u>unit</u> normal vectors $u_0, u_1, \ldots, u_n$



Without loss we can take the origin at a corner



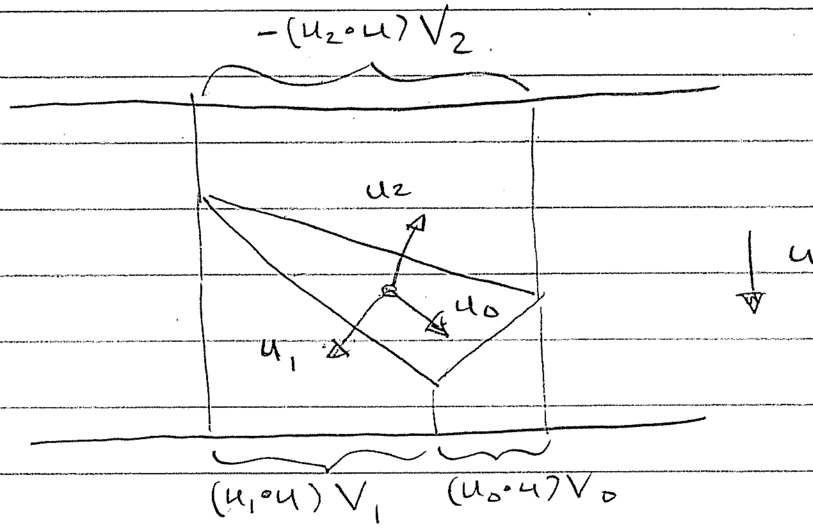
Let V_i be the $(n-1)$-dim volume of the facet $\perp$ to u_i.

Note that the vectors $u_0, \cdots, u_1$ satisfy exactly one linear relation.
In fact we have.

<u>Theorem</u>: $V_0 u_0 + V_1 u_1 + \cdots + V_n u_n = 0$.

<u>Proof</u>: Let $u \in \mathbb{R}^n$ be <u>any</u> unit vector.
Let $S = \Delta(a_1, \cdots, a_n)$ and let S_u be the orthogonal projection of S onto $u^\perp$.

For all i, note that $|u_i \cdot u| V_i$ is the volume of the projection of facet i onto $u^\perp$.



Summing over u_i such that $u_i \circ u > 0$, we obtain

$$\sum_{u_i \circ u > 0} (u_i \circ u) V_i = \mathrm{vol}(S_u).$$

Summing over u_i such that $u_i \circ u < 0$, we obtain

$$\sum_{u_i \circ u < 0} (u_i \circ u) V_i = -\mathrm{vol}(S_u).$$

Now let $w = V_0 u_0 + V_1 u_1 + \cdots + V_n u_n$ and observe that

$$\begin{aligned} w \circ u &= \sum_i (u_i \circ u) V_i \\ &= \sum_{u_i \circ u > 0} (u_i \circ u) V_i + \sum_{u_i \circ u < 0} (u_i \circ u) V_i \\ &= \mathrm{vol}(S_u) - \mathrm{vol}(S_u) \\ &= 0 \end{aligned}$$

Since this holds for all unit vectors u, we conclude that $w = 0$ $\blacksquare$

Remark: This is called the "Minkowski condition".
It holds more generally for any polytope, and it is equivalent to the divergence theorem.
(See Handout by Don Klain).

Problem: Consider the simplex $\Delta(\alpha_1, \dots, \alpha_n)$ with unit normals $u_0, u_1, \dots, u_n$.

Let $A_i = n\{\underbrace{(\alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n)}_{n-1}$

with column i deleted, and let
$A_\Delta = (\alpha_2 - \alpha_1 \quad \alpha_3 - \alpha_1 \quad \cdots \quad \alpha_n - \alpha_1)$

Then $$\sum_{i=0}^{n} u_i \frac{\sqrt{\det(A_i^t A_i)}}{(n-1)!} = 0$$

$$\sum_{i=0}^{n} u_i \sqrt{\det(A_i^t A_i)} = 0$$

Use this to give a different proof of the Minkowski condition for simplices.