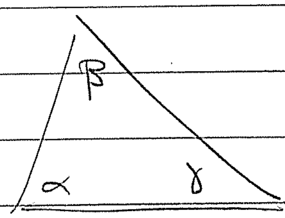
Tues Mar 26

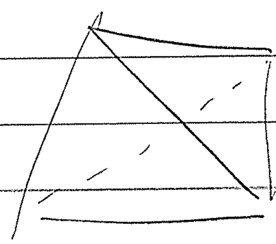
Today: "Positivity" of Forms.

Motivation: Given a Euclidean triangle



we have $\alpha+\beta+\gamma=\pi$.

Q: Given a Euclidean Tetrahedron



What algebraic relationship is satisfied by the dihedral angles θ_{ij}, $1\le i<j\le 4$?

We will answer this.

Let $GL_n(\mathbb{R})$ act on the set $Sym_n(\mathbb{R})$ of real symmetric matrices by "congruence":

$$P\circ B := PBP^t.$$

Check
$$\begin{aligned} P\circ(Q\circ B) &= P\circ QBQ^t \\ &= PQBQ^tP^t \\ &= (PQ)B(PQ)^t = (PQ)\circ B \end{aligned}$$
✓

The GL_n orbits on Sym_n are called symmetric bilinear forms, or "quadratic forms"

Recall: Sylvester's Law of Inertia (1852):

The GL_n orbits are parametrized by triples $p, m, z \in \mathbb{N}$ with $p+m+z=n$. In fact, each orbit contains a representative of the form

$$\begin{array}{c} \left(\begin{array}{c|c|c} I & & \\ \hline & I & \\ \hline & & 0 \end{array}\right) \begin{array}{l} \}\, p \\ \}\, m \\ \}\, z \end{array} \\ \quad\underbrace{\qquad}_{p}\underbrace{\qquad}_{m}\underbrace{\qquad}_{z}\qquad\qquad \end{array}$$

<u>Notation</u>: For various values we have

p	m	z	name
	0		positive semidefinite
	0	0	positive definite
0			negative semidef.
0	0		negative def.

If $p>0$ and $m>0$, the form is called "indefinite".

Note if B has signature (p, m, z) then $\exists\, P \in GL_n$ with

$$PBP^t = \begin{pmatrix} I_p & & \\ & I_m & \\ & & O_z \end{pmatrix}$$

and for any $x \in \mathbb{R}^n$ we have

$$x^t P B P^t x = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+m}^2$$

But $x^t PBP^t x = (P^t x)^t B (P^t x)$ and as x ranges over $\mathbb{R}^n \setminus 0$, so does $P^t x$. Thus we can rephrase our definition

Definition: We say $B \in \mathrm{Sym}_n(\mathbb{R})$ is pos. def. (pos. semidef.)

$$\Updownarrow$$

we have $x^t B x > 0$ (or $x^t B x \geq 0$) for all $x \neq 0$.

Recall: Congruence does <u>not</u> preserve eigenvalues. However, we still have

<u>Theorem</u>: B is pos. def. (semidef.)

$$\Updownarrow$$

All eigenvalues of B are > 0 (≥ 0).

<u>Proof</u>: First of all, we know the eigenvalues are <u>real</u> because $B^t = B$: if $Bx = \lambda x$ for some $\lambda \in \mathbb{C}$, $x \in \mathbb{C}^n$, then

$$\begin{aligned}
\lambda \|x\|^2 &= \lambda (x^t \bar{x}) \\
&= (\lambda x)^t \bar{x} \\
&= (Bx)^t \bar{x} \\
&= x^t B^t \bar{x} \\
&= x^t B \bar{x} \\
&= x^t (\overline{Bx}) \\
&= x^t \bar{\lambda} \bar{x} = \bar{\lambda} \|x\|^2
\end{aligned}$$

$\Longrightarrow \lambda = \bar{\lambda} \Longrightarrow \lambda \in \mathbb{R}$.

Now suppose that $x^t B x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

Then given any eigenvalue $Bx = \lambda x$, we have.

$$\lambda \|x\|^2 = x^t(\lambda x) = x^t B x > 0.$$

$\Longrightarrow \lambda > 0.$

Conversely, assume all eigenvalues are positive. Want to show $x^t B x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

First we show that B has an orthonormal basis of eigenvectors (weak version of the "spectral theorem").

(1) B has a (real) eigenvalue.

Proof: Given any $x \in \mathbb{R}^n$, the vectors $x, Bx, B^2x, \ldots, B^n x$ cannot be linearly independent. Hence $\exists\ a_0, a_1, \ldots, a_n \in \mathbb{R}$, not all 0, with

$$a_0 x + a_1 Bx + \cdots + a_n B^n x = 0$$
$$(a_0 I + a_1 B + \cdots + a_n B^n) x = 0.$$

By Fundamental Theorem of Algebra, this can be factored as

$$c(B-r_1I)\cdots(B-r_nI)x = 0$$

for some $c, r_1, \ldots, r_n \in \mathbb{C}$. But then some $B-r_iI$ has nontrivial kernel
$\Rightarrow r_i \in \mathbb{C}$ is an eigenvalue.
$\Rightarrow r_i \in \mathbb{R}$.

(2) If $Bu = \lambda u$, then B preserves the orthogonal hyperplane $u^\perp$

Proof: Given $v \in u^\perp$ we have

$$(Bv)^t u = v^t B^t u = v^t B u = v^t \lambda u$$
$$= \lambda v^t u = 0.$$

(3) By induction, B has an orthonormal basis of eigenvectors in $u^\perp$, say $u_1, \ldots, u_{n-1}$. Lift these trivially to $\mathbb{R}^n$ to get an O.N. eigenbasis.

$$u_1, u_2, \ldots, u_{n-1}, u_n = u.$$

Finally assume all eigenvalues of B are positive. Then for any $x \in \mathbb{R}^n$, write.

$$x = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

in the orthogonal eigenbasis. Then

$$x^t B x = (a_1 u_1^t + \cdots + a_n u_n^t)(a_1 B u_1 + \cdots + a_n B u_n)$$

$$= (a_1 u_1^t + \cdots + a_n u_n^t)(a_1 \lambda_1 u_1 + \cdots + a_n \lambda_n u_n)$$

$$= \sum_{i,j} a_i u_i^t a_j \lambda_j u_j$$

$$= \sum_{i,j} a_i a_j \lambda_j (u_i^t u_j)$$

$$= \sum_i a_i^2 \lambda_i > 0. \qquad \blacksquare$$

Thus T.F.A.E.

- B has signature $(p, 0, z)$.
- $x^t B x \geq 0 \quad \forall x$.
- all eigenvalues of B are ≥ 0.

Here's an even <u>nicer</u> characterization:

Theorem: B is positive semidefinite

$$\Updownarrow$$

$\exists$ real matrix A with $B = A^t A$.

Furthermore, we have $\ker B = \ker A$, hence the multiplicity of 0 in B is $\dim \ker A$.

<u>Proof</u>: If $B = A^t A$ then $\forall x$ we have

$$x^t B x = x^t A^t A x = (Ax)^t (Ax) = \|Ax\|^2 \geq 0.$$

Conversely, suppose B is pos. semidef. We know that B has an orthogonal eigenbasis, hence we can write

$$B = Q \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_p & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} Q^t$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p > 0$.
and $Q^t = Q^{-1}$ is orthogonal.

Now let $D = \begin{pmatrix} \sqrt{\lambda_1} & & & & & \\ & \ddots & & & & \\ & & \sqrt{\lambda_p} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$, so.

$$\begin{aligned} B &= Q D^2 Q^t \\ &= Q D D^t Q^t \\ &= QD(QD)^t \\ &= ((QD)^t)^t (QD)^t, \quad \text{as desired.} \quad \blacksquare \end{aligned}$$

Hence for real $B^t = B$, T.F.A.E.

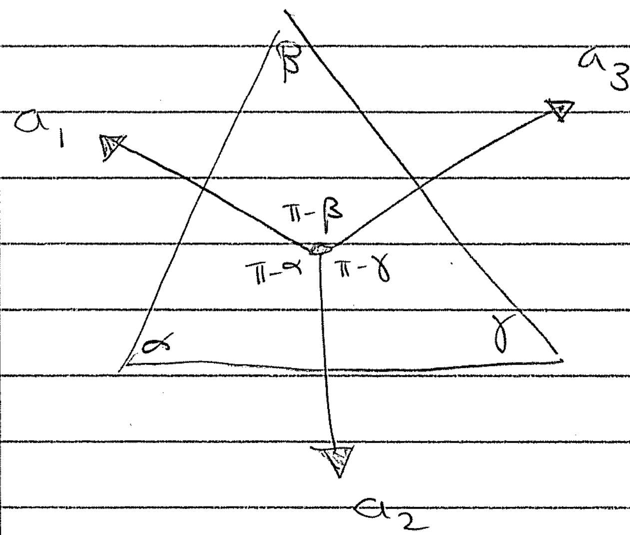
- B is pos. semidef
- $x^t B x \geq 0 \quad \forall x$
- all e.values of B are ≥ 0
- $B = A^t A$ for some real A.

Furthermore we have

$$x^t B x = 0 \iff Ax = 0$$

i.e. $\mathrm{rad}(B) = \ker(A)$.

Application to Triangles:



unit normal vectors

Let $A = (a_1\ a_2\ a_3)$. Then

$$A^t A = \begin{pmatrix} 1 & \cos(\pi-\alpha) & \cos(\pi-\beta) \\ \cos(\pi-\alpha) & 1 & \cos(\pi-\gamma) \\ \cos(\pi-\beta) & \cos(\pi-\gamma) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\cos\alpha & -\cos\beta \\ -\cos\alpha & 1 & -\cos\gamma \\ -\cos\beta & -\cos\gamma & 1 \end{pmatrix}$$

The fact that we have a triangle is equivalent to

$$\dim \ker (A^t A) = 1.$$

In particular we must have

$$\det(A^t A) = 0.$$

$$\Longrightarrow \quad 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = 0.$$

$$\Big\Updownarrow \; ?$$

$$\alpha + \beta + \gamma = \pi$$