Thurs Mar 5

We've classified FGGR's up to rank 3.

What about rank 4? and beyond?
What's the analogue of Thomas Harriot's Theorem in higher dimensions?

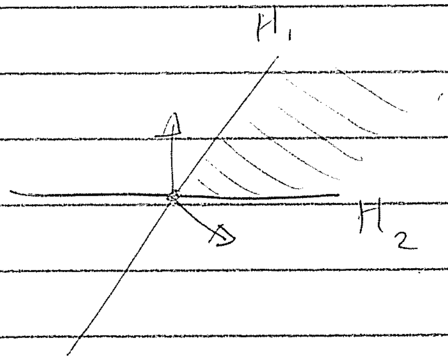
We need a bit of machinery.

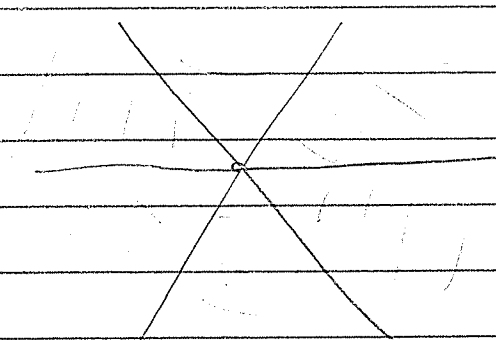
Let $G < O(n)$ be an FGGR. Then $\Sigma(G)$ decomposes $\mathbb{R}^n$ into

isometric polyhedral cones

Definition: A polyhedral cone is an intersection of closed linear half-spaces.

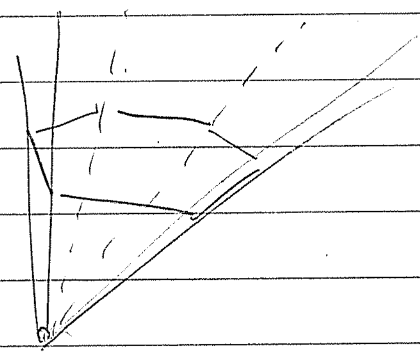
Eg





"type $G_2(3)$" has 6 polyhedral cones.

The name suggests that a polyhedral cone is equivalent to this:



"cone over a polyhedron"

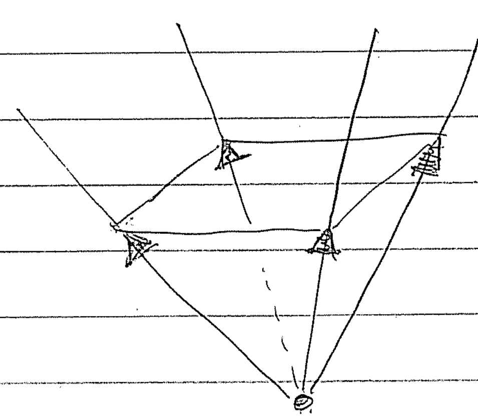
Certainly we can create a cone in this way.

Definition: Given vectors $\alpha_1, \dots, \alpha_m \in \mathbb{R}^n$ define their convex span

$$C = \{ a_1\alpha_1 + \cdots + a_m\alpha_m : a_1, \dots, a_m \geq 0 \}$$

We call this a finitely generated cone

Picture:



In general, a "cone" is any set closed under addition and non-negative scalar multiplication.

The following theorem is very tricky to prove, so we won't.

(Farkas[1898] - Minkowski[1896] - Weyl[1935])

Tricky (!) Theorem: Given a cone C,

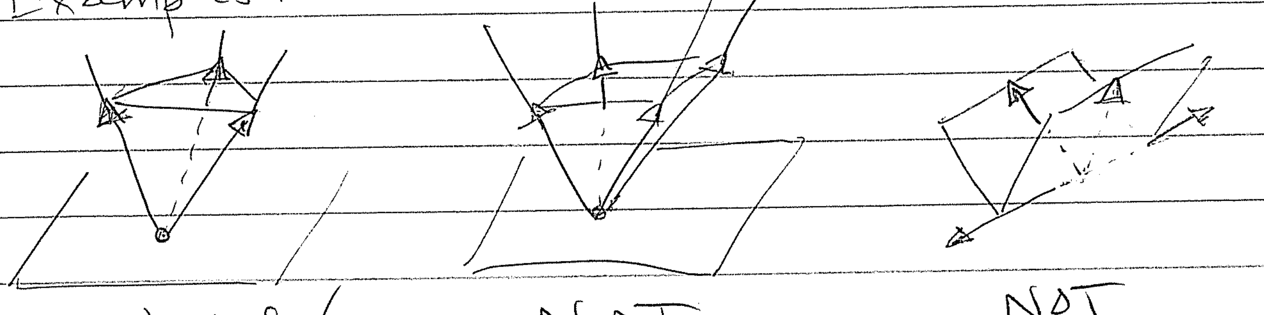
C is polyhedral $\Longleftrightarrow$ C is finitely generated.

The proof is called "Fourier - Motzkin Elimination"

However, we will prove the result for simplicial cones

Def: A cone is simplicial if it is generated by a linearly independent set.

Examples:



simplicial ✓ NOT. NOT.

Lemma: Every linearly independent set $\alpha_1, \dots, \alpha_m \in \mathbb{R}^n$ (i.e. $m \leq n$) is contained in an open half space.

Proof: Extend to a basis
$$\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_n.$$

The matrix $A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ is invertible,

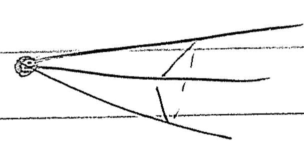
so $\exists$ vector $x \in \mathbb{R}^n$ with $\alpha_i^t x = 1 \ \forall i$.
Indeed, we have

$$Ax = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \iff x = A^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Now the set $\alpha_1, \dots, \alpha_n$ is contained in the "positive" half-space

$$H_x^+ = \{ u \in \mathbb{R}^n : u^t x > 0 \} \qquad \blacksquare$$

Corollary: Every simplicial cone is "pointed"

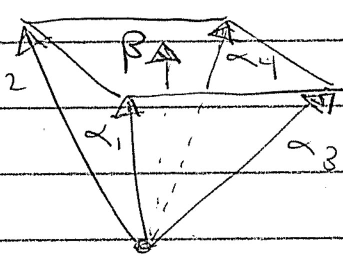


Q: What does the concept "basis" mean for a cone?

<u>Def</u>: Given a cone C we call a vector $\alpha \in C$ <u>extreme</u>, or <u>simple</u>, if α is not a convex combination of other vectors in the cone.

i.e. if $\alpha = a_1\gamma_1 + \cdots + a_m\gamma_m$ for $\gamma_1, \ldots, \gamma_m \in C$. Then $m = 1$.

<u>Picture</u>:



$\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are simple
β is not.

simple directions = "extreme rays" of the cone.

Def: A set $\alpha_1, \ldots, \alpha_m \in C$ is a simple generating set if

1. α_i generate C
2. α_i are simple
3. No α_i, α_j are collinear.

Theorem: Let pointed cone C be generated by finite set Π with no $\alpha, \beta \in \Pi$ collinear. Then Π contains a unique simple gen. set.

Proof: We will show the following. If $\alpha, \beta_1, \ldots, \beta_k \in \Pi$ generate C and α is not extreme, then $\beta_1, \ldots, \beta_k$ still generate C.

So let $\Pi = \{\alpha, \beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_\ell\}$

Since α is not extreme we can write

$$\alpha = \sum b_i \beta_i + \sum c_j \gamma_j \qquad (*)$$

with $b_i \geq 0$, $c_j \geq 0$ $\forall i,j$.

Also, since $\alpha, \beta_1, \beta_2, \ldots, \beta_k$ generate C we can write

$$\gamma_j = d_j \alpha + \sum_i f_{ji} \beta_i \qquad (**)$$

with $d_j \geq 0$ and $f_{ji} \geq 0 \quad \forall\, i, j$.
Substituting $(**)$ into $(*)$ gives

$$\alpha = \sum_i b_i \beta_i + \sum_j c_j \left(d_j \alpha + \sum_i f_{ji} \beta_i \right)$$

$$= \sum_i \left(b_i + \sum_j c_j f_{ji} \right) \beta_i + \alpha \left(\sum_j c_j d_j \right)$$

$$(***) \qquad \alpha \left(1 - \sum_j c_j d_j \right) = \sum_i \underbrace{\left(b_i + \sum c_j f_{ji} \right)}_{\geq 0} \beta_i \in C.$$

Since $(1 - \sum_j c_j d_j)\alpha \in C$ we have

$$1 - \sum_j c_j d_j \geq 0$$

But if $1 - \sum_j c_j d_j = 0$ we get

$$\sum_i \left(b_i + \sum_j c_j f_{ji} \right) \beta_i = 0,$$

which contractiicts the fact that C is <u>pointed</u>.

Hence $1 - \sum_j c_j d_j > 0$. From (***) we get

$$\alpha = \frac{1}{1 - \sum_j c_j d_j} \sum_i \left(b_i + \sum_j c_j f_{ji} \right) \beta_i$$

Since $\alpha, \beta_1, \ldots, \beta_k$ generate C,
we conclude that $\beta_1, \ldots, \beta_k$ generate C

Summary: Every finite generating set for a pointed cone contains a unique "basis".
The basis points along the "extreme rays".

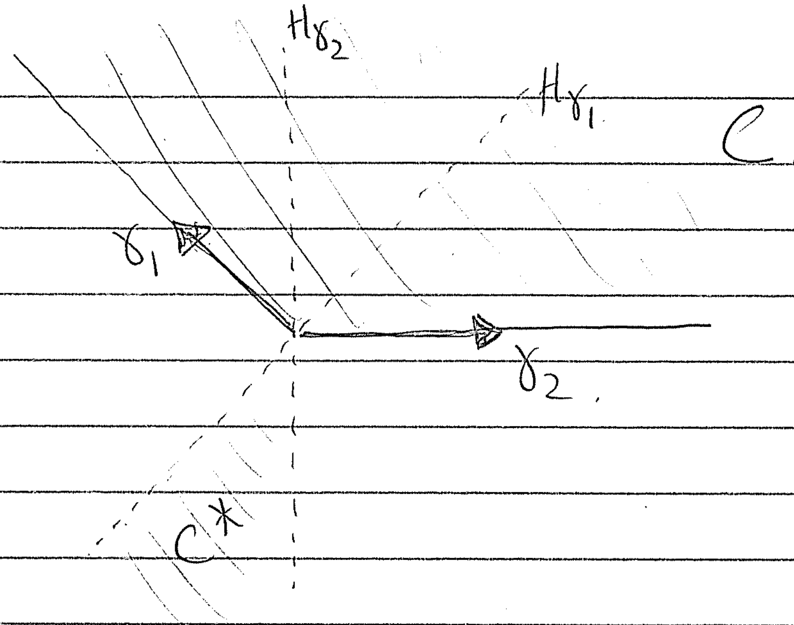
Duality for Cones.

Given a cone $C \subseteq \mathbb{R}^n$ define its <u>dual</u> (or <u>polar</u>)

$$C^* = \{ x \in \mathbb{R}^n : x^t y \leq 0 \ \forall y \in C \}$$

Clearly if C is finitely generated then C^* is polyhedral

Picture:



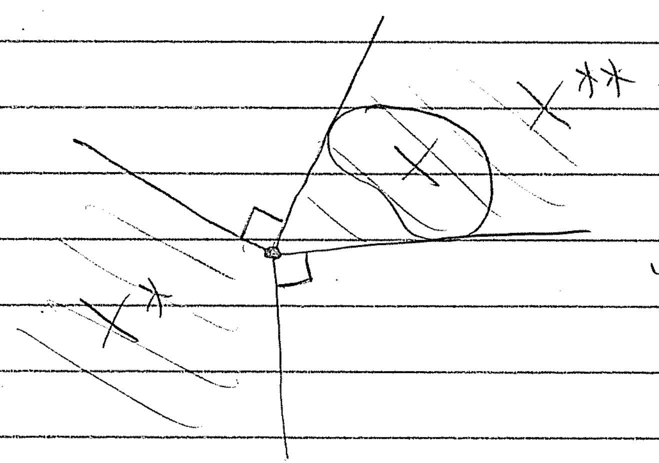
For general set $X \subseteq \mathbb{R}^n$ we can define

$$X^* = \{ x \in \mathbb{R}^n : x^t \gamma \leq 0 \ \forall \gamma \in X \}.$$

One can verify (Exercise)

1. X^* is convex
2. $X^{**} \supseteq X$.
3. X^{**} is the smallest convex cone containing X.

Picture



"Galois"

Finally we can state

The Duality Theorem for Cones:

Let C be a finitely generated cone. Then C^* is also finitely generated, and it follows that

$$C^{**} = C$$

Corollary: Let C be a cone. Then

C is f.g. $\iff$ C is polyhedral

Again: The general proof is tricky.

Next time we will prove the simplicial case.