Thurs Feb 21

Recall: The finite subgroups of $SO(3)$ are

	C_n,	D_{2n},	T,	O,	I
order	n,	$2n$,	12,	24,	120.

The finite subgroups of $O(3)$ fall into three classes

(1) Subgroups of $SO(3)$

C_n, D_{2n}, T, O, I

(2) Groups containing -1.

$C_n \cup -C_n$, $D_{2n} \cup -D_{2n}$, $T \cup -T$, $O \cup -O$, $I \cup -I$.

(3) Groups not containing -1.
"Mixed Types"

$C_n \cup -(C_{2n} \setminus C_n)$
$C_n \cup -(D_{2n} \setminus C_n)$
$D_{2n} \cup -(D_{4n} \setminus D_{2n})$
$T \cup -(O \setminus T)$.

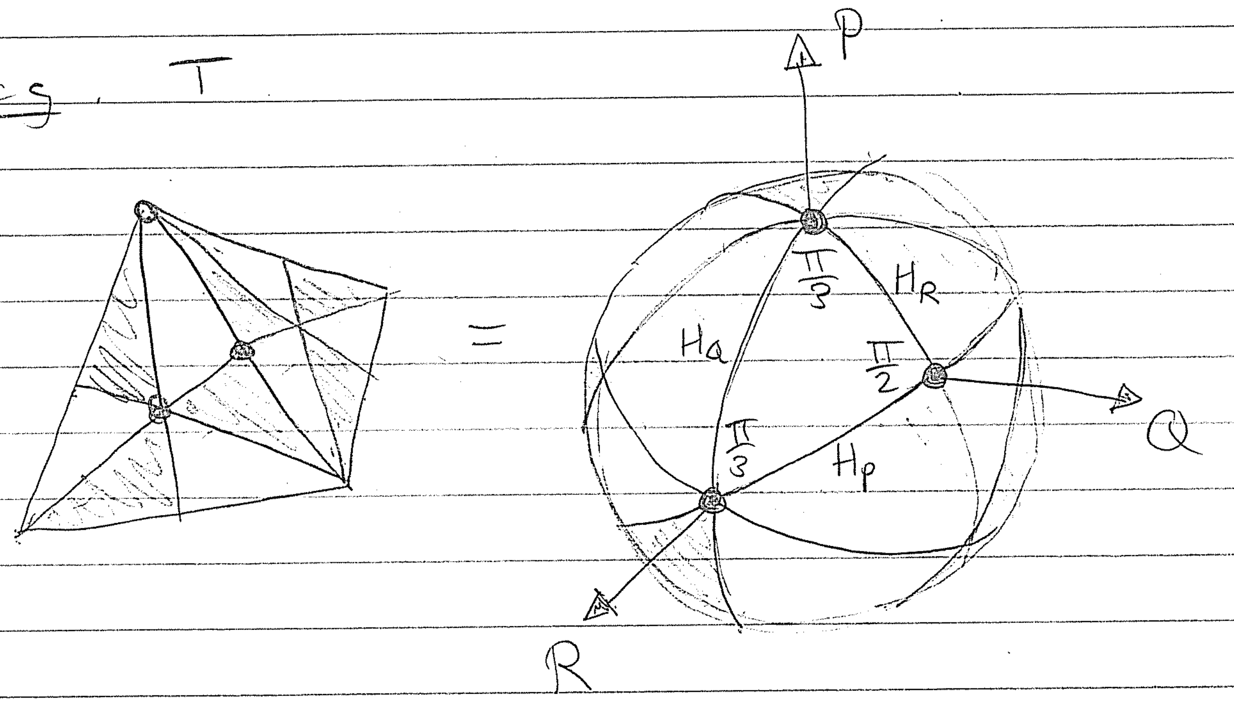
That's All. Total: 14 kinds.
(7 families, 7 exceptional)

We isolate 6 special groups..

Shape	$\mathrm{Aut}^+(\text{Shape})$	$\mathrm{Aut}(\text{Shape})$
Tetrahedron	T	$T \cup -(O \setminus T)$
Cube / Octahedron	O	$O \cup -O$
Dodec / Icosahedron	I	$I \cup -I$

Recall: T, O, I are generated by rotations around the vertices of a spherical triangle

<u>Eg</u>. T



$$T = \left\langle \mathrm{Rot}_{\frac{2\pi}{3}}(P),\ \mathrm{Rot}_{\pi}(Q),\ \mathrm{Rot}_{\frac{2\pi}{3}}(R) \right\rangle$$

Recall:

$$\mathrm{Rot}_{\frac{2\pi}{3}}(P) = \mathrm{Ref}(H_Q)\,\mathrm{Ref}(H_R)$$

$$\mathrm{Rot}_{\pi}(Q) = \mathrm{Ref}(H_R)\,\mathrm{Ref}(H_P)$$

$$\mathrm{Rot}_{\frac{2\pi}{3}}(R) = \mathrm{Ref}(H_P)\,\mathrm{Ref}(H_Q).$$

Hence (as Euler knew)

$$\mathrm{Rot}_{\frac{2\pi}{3}}(P)\,\mathrm{Rot}_{\pi}(Q)\,\mathrm{Ref}_{\frac{2\pi}{3}}(P) = I.$$

The full automorphism group is generated by the <u>reflections</u>.

$$T\text{-}(\triangle T) = \langle \underbrace{\mathrm{Ref}(H_P)}_{s_1}, \underbrace{\mathrm{Ref}(H_Q)}_{s_2}, \underbrace{\mathrm{Ref}(H_R)}_{s_3} \rangle.$$

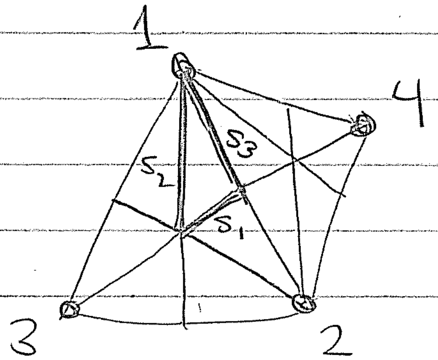
It has abstract presentation

$$\langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = 1 \rangle$$

Which can be displayed schematically:

$$\begin{tikzcd} s_1 \arrow[r, dash, "3"] \arrow[rr, dash, bend right, "2"'] & s_2 \arrow[r, dash, "3"] & s_3 \end{tikzcd} \quad \overset{\text{"}\ \text{"}}{=} \quad \begin{tikzcd} s_1 \arrow[r, dash] & s_2 \arrow[r, dash] & s_3 \end{tikzcd}$$

Note: This is just the group of permutations of the vertices:



$s_1 = (12)$
$s_2 = (23)$
$s_3 = (34)$

[Remark: In general, the group $\mathfrak{S}_n$ of permutations of $\{1, 2, \ldots, n\}$ is generated by the adjacent transpositions

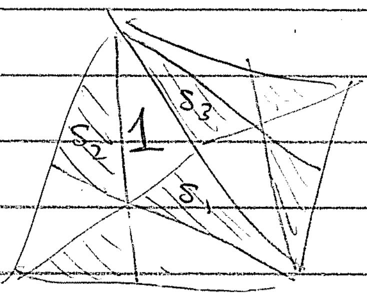
$$s_i = (i, i+1)$$

with presentation

$$\left\langle s_1, s_2, \ldots, s_{n-1} : \begin{array}{l} s_i^2 = 1 \quad \forall\, i \\ (s_i s_j)^3 = 1 \quad \forall\, |i-j| = 1 \\ (s_i s_j)^2 = 1 \quad \forall\, |i-j| > 1 \end{array} \right\rangle$$

This is the full symmetry group of the regular hypersimplex in $\mathbb{R}^{n-1}$]

Finally, note that $T \cup -(O \setminus T)$ acts regularly on the chambers of the barycentric subdivision:



T = white triangles

$-O \setminus T$ = black triangles

The convex hull of the orbit of a point has 24 vertices. It is called the "permutohedron". Its vertices are the permutations $= \mathfrak{S}_4$.

We can also define a "length function" $\ell : \mathfrak{S}_4 \to \mathbb{N}$ and a "distance"

$d : \mathfrak{S}_4 \times \mathfrak{S}_4 \to \mathbb{N}$.

$d(g,h) = \ell(gh^{-1}) = \#$ (hyper)planes separating chambers g and h.

The "longest" permutation is

$$1\,2\,3\,4 = (14)(23)$$
$$= s_1 s_2 s_3 s_1 s_2 s_1$$

with length = total # reflections = 6.

What are the reflections in $\mathfrak{S}_4$?

$$\underbrace{\underbrace{(12),\ (23),\ (34)}_{\text{adjacent}},\ (13),\ (24),\ (14)}_{\text{all transpositions}}.$$

Define the length generating function

$$\mathfrak{S}_4(q) = \sum_{\pi \in \mathfrak{S}_4} q^{\ell(\pi)}$$

$$= 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + 1q^6$$

$$= 1\cdot(1+q)(1+q+q^2)(1+q+q^2+q^3)$$

$$= [4]_q! \quad \text{“} q\text{-factorial”}$$

Yes: In general we have

$$\mathfrak{S}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\ell(\pi)} = [n]_q!$$

with maximum length

$\binom{n}{2}$ = # transpositions / reflections.

In fact, the whole discussion generalizes to

Groups Generated by Reflections

Let FGGR = "Finite Group Generated by Reflections"

BEGIN

Let $G < O(n)$ be an FGGR with set of reflections

(Think: "transpositions")

$T = \{t_1, t_2, \dots, t_N\}$

Note: For all $g \in G$, $t \in T$ we have $gtg^{-1} \in T$

Let $t_i \in T$ have reflecting hyperplane H_{t_i}. Then for all $g \in G$ we have

$$\boxed{g(H_{t_i}) = H_{g t_i g^{-1}}}$$

Now let general hyperplane $H \subseteq \mathbb{R}^n$ have reflection $t_H \in O(n)$, so $t_{H_t} = t$ and $H_{t_H} = H$.

Bijection: Reflections $\leftrightarrow$ Hyperplanes

Now consider the collection of reflecting hyperplanes of G:

$$\Sigma(G) = \{H_{t_1}, H_{t_2}, \dots, H_{t_N}\}$$
(an "arrangement" of hyperplanes)

Theorem: $\Sigma(G)$ is a closed mirror system (CMS) in the sense that

$$\forall\, H_i, H_j \in \Sigma(G) \text{ we have } t_{H_i}(H_j) \in \Sigma(G)$$

Proof: Given $H_{t_1}, H_{t_2} \in \Sigma(G)$ we have

$$t_{H_{t_i}}(H_{t_j}) = t_i(H_{t_j}) = H_{t_i t_j t_i^{-1}} \in \Sigma(G)$$

because $t_i t_j t_i^{-1} \in T$ $\boxed{\text{W}}$

Conversely, let Σ be any CMS and consider the group generated by its reflections

$$G(\Sigma) = \langle t_H : H \in \Sigma \rangle$$

Theorem: $G(\Sigma)$ is an FGGR.

Proof: Certainly it's a GGR. We must show that it's finite.

More generally, we will show: If group G is generated by finite set T of involutions such that

$\forall\, t_1, t_2 \in T$ we have $t_1 t_2 t_1^{-1} (= t_1 t_2 t_1) \in T$,

then G is finite.

Indeed, given $g \in G$ we write $g = t_1 t_2 \cdots t_k$ where $t_i \in T$ and k is minimal. Then this word contains no repeated reflection since otherwise

$$g = t_1 \cdots t_i\, t\, t_{i+1} \cdots t_j\, t\, t_{j+1} \cdots t_k$$

$$= \underbrace{t_1 \cdots t_i (t t_{i+1} t) \cdots (t t_j t) t_{j+1} \cdots t_k}_{k-2 \text{ reflections}},$$

contradicting the minimality of k.

Since every $g \in G$ can be written as a minimal word in T with no repeated letters, we have

$$|G| \leq \underbrace{1 + |T| + |T|^2 + \cdots + |T|^{|T|}}_{\#\text{words of length} \leq |T|} < \infty$$

$\boxtimes$

We obtain a bijective correspondence

$$\text{FGGR's} \longleftrightarrow \text{FCMS's}$$

$$G \longrightarrow \Sigma(G)$$

$$G(\Sigma) \longleftarrow \Sigma$$

Recall that any finite $G < O(n)$ has a fixed point $g(x) = x \ \forall g \in G$.
If G is an FGGR then every reflection t fixes x, so every hyperplane H_t contains x.

Conclusion: We may assume that all the hyperplanes are linear (i.e. contain 0).