Tues Feb 19

Part IV: Reflection Groups.

We've seen two examples of ADE classification. Now I'll show you a <u>third</u>.

Motivation:

$$\begin{array}{ccc} SU(2) & & O(3) \\ & \searrow & \swarrow \\ & SO(3) & \end{array}$$

We lifted $D_{p,q,r} < SO(3)$ to $D^*_{p,q,r} < SU(2)$.

Can we lift $D_{p,q,r}$ to $O(3)$?

Recall the geometry of $O(3)$.
(by Cartan–Dieudonné)

# reflections	geometry
0	id
1	reflection
2	rotation
3	screw reflection.

That's All.

Consider the "absolute value" hom

$$O(3) \xrightarrow{\text{abs}} SO(3)$$
$$A \longmapsto \det(A) \cdot A.$$

The kernel is $\{\pm 1\}$, so

$$SO(3) = O(3) / \{\pm 1\}.$$

But this time $O(3) = SO(3) \times \{\pm 1\}$

direct product, boring.

Q: Finite subgroups of $O(3)$?

Let $G < O(3)$ be finite. There are 3 cases.

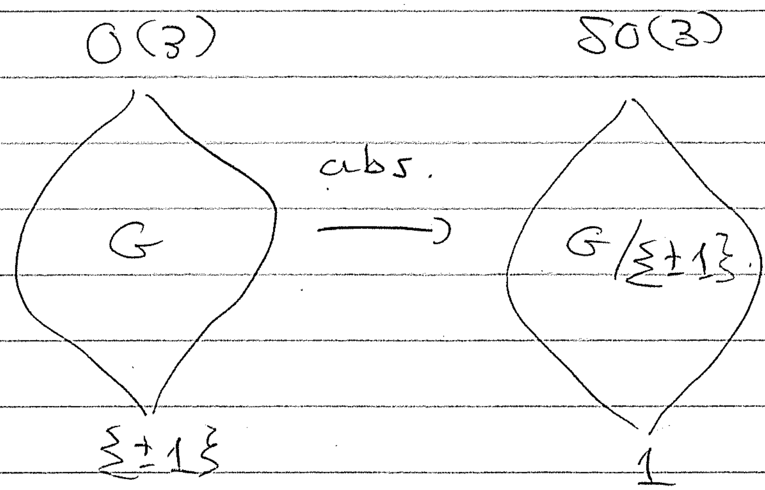
① $G < SO(3)$, so $G = D_{p,q,r}$.

② $-1 \in G$.

Let $G^+ = G \cap SO(3)$, so $G^+ = D_{p,q,r}$.

Claim: $G = G^+ \times \{\pm 1\}$

Indeed, we have a correspondence.



(3) $-1 \notin G$.

Let $G^+ = G \cap SO(3) = \{S_1, S_2, \dots, S_n\}$ and choose any $R \in G \cap O(3) \setminus SO(3)$.

Let $R_i = RS_i$. Then we have
$G \cap O(3) \setminus SO(3) = \{R_1, R_2, \dots, R_n\}$.

Let $T_i = -R_i \in SO(3)$.

Note $\forall\, i, j$ that

$$T_i T_j = (-R_i)(-R_j) = R_i R_j \in G^+$$
$$S_i T_j = S_i(-R_j) = -S_i R_j \in G^+$$

Thus $K = \{S_1, S_2, \dots, S_n, T_1, \dots, T_n\} < SO(3)$.

So $G^+ < K < SO(3)$ and we can write

$$G = G^+ \cup -(K \setminus G^+).$$

Conversely, given any pair

$$G^+ < K < SO(3) \text{ with } [K : G^+] = 2,$$

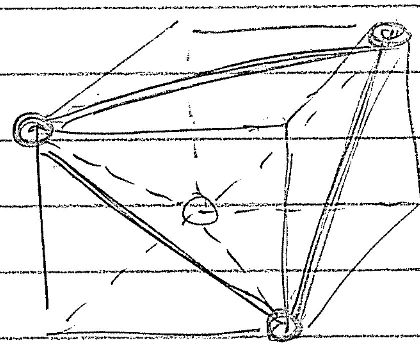
we can define the group

$$G = G^+ \cup -(K \setminus G^+) < O(3).$$

To complete the classification, we must classify such pairs $G^+ < K$.

Omitted. ∎

There is really just one interesting case:



$$\overset{12}{T} < \overset{24}{O}$$

index 2.

Note: $T \cup -(O \setminus T)$ is the full symmetry group of the tetrahedron.

T = proper symmetries of tetrahedron.

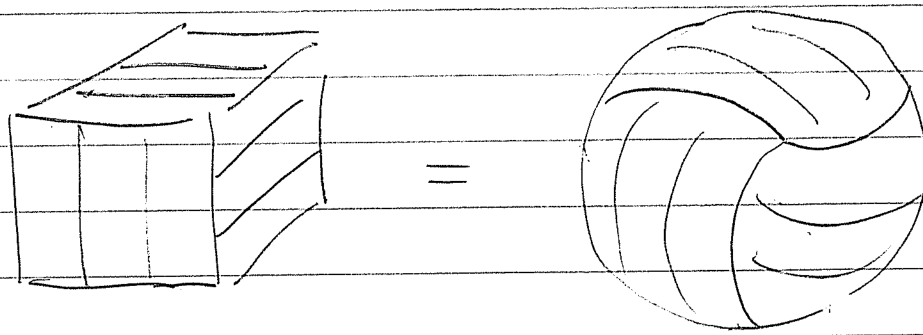
$-O \setminus T$ = improper symmetries of tetrahedron.

[Remark: The easier group

$$T \cup -T = T \times \{\pm 1\}$$

is called the "pyritohedral group"

= symmetries of a volleyball



]

So... how to "lift" $D_{p,q,r}$ to $O(3)$.

Two choices.

(1)

$T \to T \cup -T$ volleyball

$O \to O \cup -O$ octahedron

$I \to I \cup -I$ icosahedron

OR

(2)

$T \to T \cup -(O \setminus T)$ tetrahedron

$O \to O \cup -O$ octahedron

$I \to I \cup -I$ icosahedron.

which is better?

We choose (2), because.

- regular polyhedra are more natural than volleyballs.
- it generalizes to <u>all dimensions</u>. (and beyond)

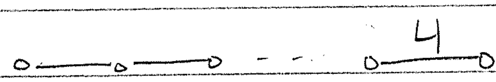
I'll give away the secret right now.

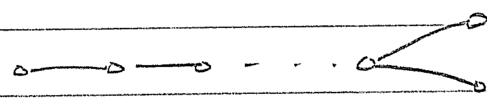
Let $G < O(n)$ be finite and generated by reflections, then

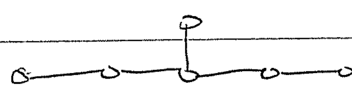
Theorem (Coxeter, 1935):

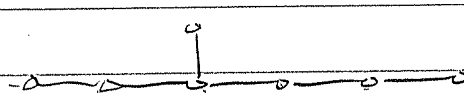
G corresponds to a Coxeter diagram.

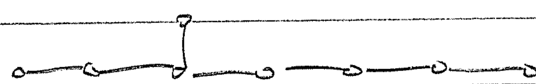
A_n  $(n \geq 1)$

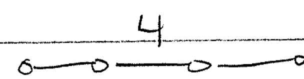
B_n  $(n \geq 2)$

D_n  $(n \geq 4)$.

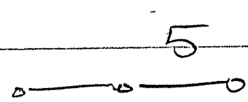
E_6 

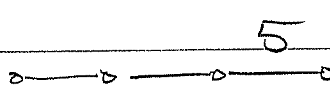
E_7 

E_8 

F_4 

$G_2(m)$ 

H_3 

H_4 

That's All.

So far we've seen

$G_2(m) = $ dihedral order $2m$ } 2-dim

$A_3 = TU-(O \setminus T)$ order 24
$B_3 = OU - O$ order 48
$H_3 = IU - I$ order 120
} 3-dim

We've glimpsed F_4 and H_4

Let me explain: . .

BEGIN.

Let $G < O(n)$ be finite and generated by reflections.

Let $T = \{t_1, t_2, \dots, t_m\} \subseteq G$ be the set of reflections and let

$$H_{t_i} = \ker(1 - t_i) \subseteq \mathbb{R}^n$$

be the reflecting hyperplanes.

Note that $G \curvearrowright T$ by conjugation:

Given $g \in G$, $t \in T$ we have $gtg^{-1} \in T$.

<u>Proof</u>:
A reflection is an orthogonal map with eigenvalues $-1, 1, \ldots, 1$. Note that t, gtg^{-1} are orth. with same e.values ///.

In fact, given $x \in \mathbb{R}^n$, note that

$$t(x) = x \implies gtg^{-1}(g(x)) = gt(x)$$
$$= g(t(x))$$
$$= g(x).$$

and. $$gtg^{-1}(g(x)) = g(x) \implies g(t(x)) = g(x).$$
$$\implies g^{-1}g(t(x)) = g^{-1}g(x)$$
$$\implies t(x) = x.$$

We conclude that

$$\boxed{H_{gtg^{-1}} = g(H_t)}$$

Given a hyperplane $H \subseteq \mathbb{R}^n$, let $t_H \in O(n)$ be its reflection.

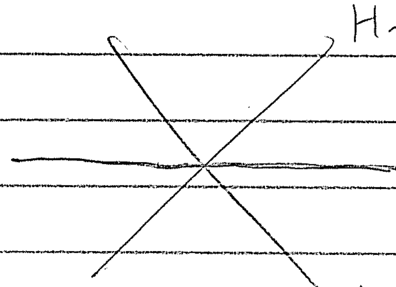
Definition: We say hyperplanes in $\mathbb{R}^n$

$$\Sigma = \{H_1, H_2, \ldots, H_m\}$$

form a closed mirror system if

$\forall i, j$ we have $t_{H_i}(H_j) \in \Sigma$.

Example:



$H_2 = t_{H_1}(H_3) = t_{H_3}(H_2)$.

$H_1 = t_{H_2}(H_3) = t_{H_3}(H_2)$

$H_3 = t_{H_1}(H_2) = t_{H_2}(H_1)$

If $G < O(n)$ is generated by its reflections $T = \{t_1, \ldots, t_m\} \subseteq G$, then

$$\Sigma(G) = \{H_{t_1}, \ldots, H_{t_m}\} \text{ is a CMS}$$

because $t_{H_{t_i}}(H_{t_j}) = t_i H_{t_j} = H_{t_i t_j t_i^{-1}} \in \Sigma(G)$

//.