Tues Jan 15
2013

# Reflection Groups Part 2

## Prologue / Review

Let $V$ be a vector space over $\mathbb{F}$ with char $\mathbb{F} \neq 2$ and $\dim V = n$.
Let $B : V \times V \to \mathbb{F}$ be a non-degenerate, symmetric bilinear form.
We say $\varphi : V \to V$ is a $B$-isometry if $\forall\, x, y \in V$ we have

$$B(\varphi(x) - \varphi(y), \varphi(x) - \varphi(y)) = B(x-y, x-y)$$

$$\text{"} \|\varphi(x) - \varphi(y)\|^2 = \|x-y\|^2 \text{"}$$

Let $\mathrm{Isom}(\mathbb{A}V, B)$ be the group of isometries. Recall

$$\mathrm{Isom}(\mathbb{A}V, B) = V \rtimes O(V, B)$$

where

$$O(V, B) = \{ B\text{-isometries } \varphi : V \to V \text{ such that } \varphi(0) = 0 \}.$$

Theorem

$= \mathrm{Isom}(AV, B) \cap GL(V)$

$= \{$ invertible matrices $A$ such that $(Ax)^t[B](Ax) = x^t[B]x \quad \forall x \in V \}$.

[Recall the Gram matrix $[B] = [B(e_i, e_j)]_{i,j}$.
Then we have

$$[B(u,v)] = [u]^t[B][v].$$ ]

So $(Ax)^t[B](Ax) = x^t[B]x \quad \forall x$

$$\Longrightarrow x^t(A^t[B]A)x = x^t[B]x \quad \forall x.$$

$$\Longrightarrow \boxed{A^t[B]A = [B]}$$

We say the matrix $A$ is "B-orthogonal".

Example: Dot product $[B] = I$

$$\Longrightarrow A^tA = I.$$

Theorem (Cartan-Dieudonné):

If $A \in O(V, B)$ then $\exists$ reflections $R_1, R_2, \ldots, R_k$ with $k \leq n$ such that

$$A = R_1 R_2 \cdots R_k.$$

[DEF: A B-reflection is a B-orthogonal matrix $R \in O(V, B)$ such that $\ker R = x^\perp$ for some $B(x, x) \neq 0$ ]

Corollary: If $\varphi \in \mathrm{Isom}(AV, B)$ then $\exists$ affine reflections $R_1', R_2', \ldots, R_k'$ with $k \leq n + 1$ such that

$$\varphi = R_1' \circ R_2' \circ \cdots \circ R_k'$$

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Now consider a degree 2 field extension $\mathbb{F} \subseteq \mathbb{F}[\alpha]$ (i.e. $\alpha \notin \mathbb{F}$, $\alpha^2 \in \mathbb{F}$). with conjugation

$$\overline{a + b\alpha} = a - b\alpha .$$

Note that

$$\underbrace{\overline{(a+b\alpha)}(a+b\alpha)}_{\text{"}\|a+b\alpha\|^2\text{"}} = a^2 - b^2\alpha^2 \in \mathbb{F}$$

This motivates the following definition: We say $B : V/\mathbb{F}[\alpha] \times V/\mathbb{F}[\alpha] \longrightarrow \mathbb{F}[\alpha]$ is sesquilinear (or hermitian) if

① $B(\circ, x) : V \to V$ is linear $\forall x \in V$

② $B(x,y) = \overline{B(y,x)}$ $\forall\, x,y \in V$

The Gram matrix $[B] = [B(e_i, e_j)]$ is also called hermitian:

$$\overline{[B]}^{t} = [B]$$

In coordinates we have

$$B(x,y) = \bar{x}^t [B]\, y$$

Nice Feature:

$$\begin{aligned} B(x,x) &= \bar{x}^t [B]\, x \\ &= \bar{x}^t\, \overline{[B]}^t\, x \\ &= \overline{\left(\bar{x}^t [B]\, x\right)}^t \\ &= \overline{B(x,x)}^t = \overline{B(x,x)} \end{aligned}$$

$$\Longrightarrow B(x,x) \in \mathbb{F}$$

That's good.

We can think: $B(x,x) =$ "$\|x\|^2$"

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DEF: We say $A \in \mathrm{Mat}_n(\mathbb{F}[\alpha])$ is $B$-unitary if

$$\boxed{\bar{A}^t [B]\, A = [B]}$$

Take determinants:

$$\det(\bar{A}^t [B] A) = \det([B])$$

$$\det(\bar{A}^t)\, \cancel{\det([B])}\, \det(A) = \cancel{\det([B])}$$

If $B$ is non-degenerate, divide to get

$$\det(\bar{A}^t) \det(A) = 1$$

$$\overline{\det(A)} \det(A) = 1$$

$$\text{"}\,\|\det(A)\|^2\,\text{"} = 1$$

i.e. unitary

Finally, we can restrict a hermitian form

$$\begin{array}{ccc} \mathbb{F}[\alpha] & & B \text{ hermitian} \\ \downarrow & & \downarrow \\ \mathbb{F} & & B \text{ symmetric}. \end{array}$$

End of Prologue
BEGIN COURSE.

Let $\mathbb{F} = \mathbb{R}$, $\alpha = \sqrt{-1}$, $\mathbb{F}[\alpha] = \mathbb{C}$.

Consider the <u>standard</u> hermition form on $\mathbb{C}^n$

$$\langle x, y \rangle = \overline{x}_1 y_1 + \overline{x}_2 y_2 + \cdots + \overline{x}_n y_n$$

which restricts to dot product on $\mathbb{R}^n$.

Let

$$O(n) = \{ A \in \mathrm{Mat}_n(\mathbb{R}) : A^t A = I \}$$
$$U(n) = \{ A \in \mathrm{Mat}_n(\mathbb{C}) : \overline{A}^t A = I \}$$

Note. $A \in O(n) \Rightarrow \det(A) = \pm 1$.
$A \in U(n) \Rightarrow |\det(A)| = 1$.

Define

$$SO(n) = \{ A \in O(n) : \det(A) = +1 \}$$
$$SU(n) = \{ A \in U(n) : \det(A) = +1 \}.$$

Start Small : $O(1) = \{\pm 1\}$, $SO(1) = \{1\}$,

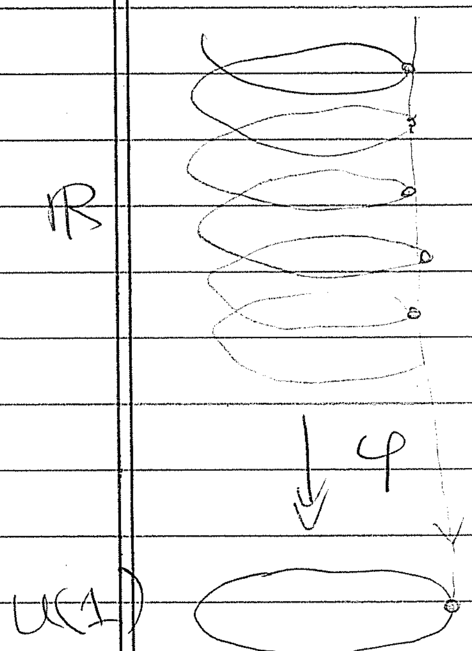
$$U(1) = \{ z \in \mathbb{C} : |z| = 1 \}$$
$$= \{ e^{i\theta} : \theta \in \mathbb{R} \}$$

Get a nice surjective morphism

"Lie algebra" $\qquad$ "Lie group"

$$(\mathbb{R}, +) \xrightarrow{\varphi} U(1)$$
$$\theta \longmapsto e^{i\theta}$$

Also a topological covering map.




the "universal" covering space,

topologically a circle $S^1$.

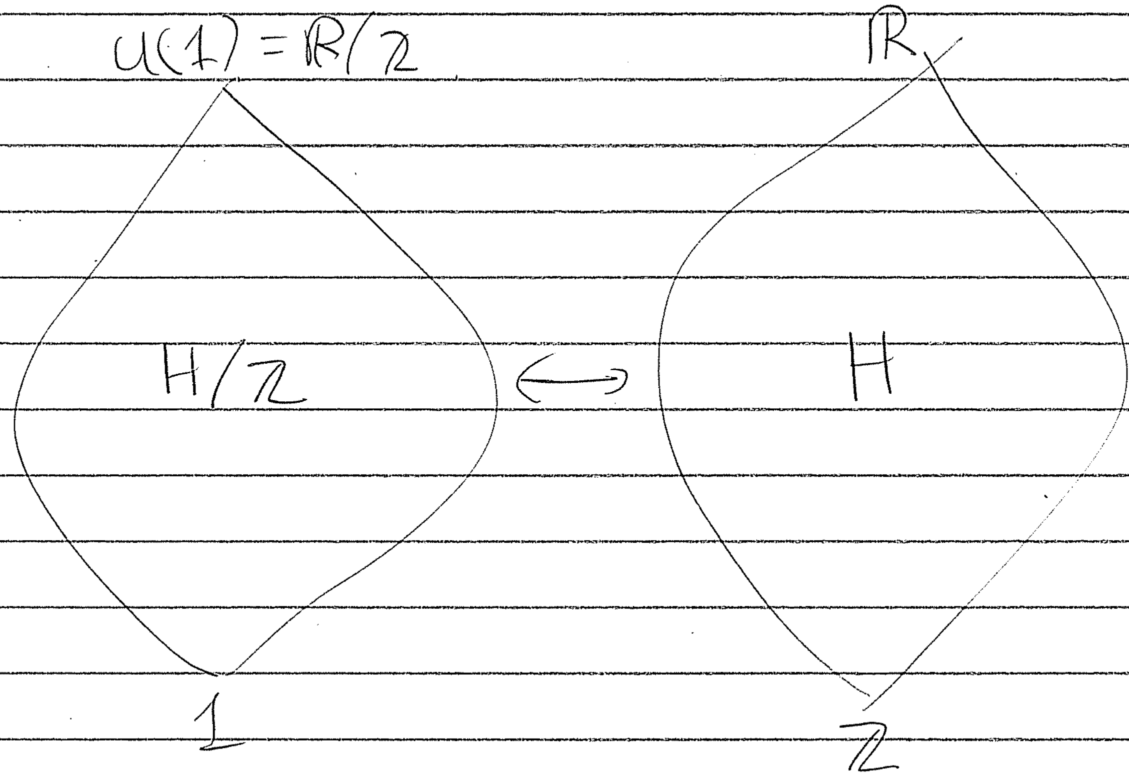
The kernel of $\varphi$ (i.e. the fiber $\varphi^{-1}(1)$) is $2\pi\mathbb{Z}$. Hence

$$\mathcal{U}(1) \approx \mathbb{R}/2\pi\mathbb{Z} \approx \mathbb{R}/\mathbb{Z}.$$

Q: <u>Discrete</u> subgroups of $\mathcal{U}(1)$?

Correspondence:

Disc. subgroups of $\mathcal{U}(1)$ $\longleftrightarrow$ Discrete subgroups of $\mathbb{R}$, containing $\mathbb{Z}$



Lemma: Let $H < \mathbb{R}$ be discrete. Then $H = \{0\}$ or $H = x\mathbb{Z}$ for some $x > 0$.

Proof: Let $H < \mathbb{R}$ be discrete, so $\exists\, \varepsilon > 0$ with $|x-y| > \varepsilon$ $\forall\, x, y \in H$.

Suppose $H \neq \{0\}$ and choose $z \neq 0$ in $H$. Since $H$ is a group, also have $-z \in H$. Without loss, say $z > 0$.

The set $H \cap (0, z]$ is finite, hence has a smallest elt $x$. This $x$ is the smallest pos elt of $H$.

Claim: $H = x\mathbb{Z}$
Indeed, $x\mathbb{Z} < H$. Conversely, given any $y \in H$, let $y = rx$.
Write $r = m + r_0$ where $m \in \mathbb{Z}$ and $0 \leq r_0 < 1$. But then

$$0 \leq r_0 < 1.$$
$$0 \leq r - m < 1.$$
$$0 \leq rx - mx < x$$
$$0 \leq y - mx < x$$

Since $y - mx \in H$, we conclude that
$y - mx = 0 \implies y \in x\mathbb{Z}$ $\boxtimes$

Thus,

$$\{\text{disc. sbgps of } \mathbb{R}, \text{ containing } \mathbb{Z}\}$$

$$= \{\text{disc. sbgps of } \mathbb{R} \text{ containing } 1\}$$

$$= \{x\mathbb{Z} \text{ where } xa = 1 \text{ for some } a \in \mathbb{Z}\}$$

$$= \{\tfrac{1}{a}\mathbb{Z} \ : \ a \in \mathbb{Z} \setminus \{0\}\}.$$

We conclude that the discrete subgroups of $U(1)$ are

$$\approx \underbrace{\frac{\frac{1}{a}\mathbb{Z}}{\mathbb{Z}}} \approx \underbrace{\mathbb{Z}/a\mathbb{Z}}_{\text{cyclic}} \quad \text{for } a \in \mathbb{Z} \setminus \{0\}.$$

Explicitly these are ath roots of 1:

$$U_a := \{e^{2\pi i k/a} : k = 1, 2, \ldots, a\}.$$

<u>That's all</u>