Thurs, Sept 6

Q: What is a "matrix"?

Answer 1: An endomorphism of a finite dimensional vector space, expressed in coordinates

Given $\varphi \in End(V, \mathbb{R})$ and basis $\beta \subseteq V$, $\beta = \{\vec{e}_1, \ldots, \vec{e}_n\}$, φ is determined by an n-tuple of n-tuples

$$[\varphi]_\beta := \left[[\varphi(\vec{e}_1)]_\beta \ [\varphi(\vec{e}_2)]_\beta \ \cdots \ [\varphi(\vec{e}_n)]_\beta \right]$$

called a "matrix". This implicitly defines an "algebra" structure on matrices

$$(End(V, \mathbb{F}), +, \circ) \xrightarrow[\beta]{\sim} (Mat(n, \mathbb{F}), +, \times)$$

$$\varphi \longmapsto [\varphi]_\beta$$

Given 2 bases β and $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ $\exists$ a transition matrix $C_{\alpha\beta}$, $\forall \vec{x} \in V$

$$[\vec{x}]_\beta = C_{\alpha\beta} [\vec{x}]_\alpha \quad \& \quad [\vec{x}]_\alpha = C_{\alpha\beta}^{-1} [\vec{x}]_\beta.$$

Exercise:

$$C_{\alpha\beta} = \left[[\vec{v}_1]_\beta \ [\vec{v}_2]_\beta \ \cdots \ [\vec{v}_n]_\beta \right]$$

It follows that $\forall\ \varphi \in \mathrm{End}$,

$$\boxed{[\varphi]_\beta C_{\alpha\beta} = C_{\alpha\beta} [\varphi]_\alpha}.$$

☆ Big Idea: Given $\varphi \in \mathrm{End}(V)$ choose basis so $[\varphi]_\beta$ is nice.
(i.e. Given matrix $A \in \mathrm{Mat}(n)$ choose invertible matrix C so CAC^{-1} is nice.)

(1) Best: If A has an eigenbasis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ with $A\vec{v}_i = \lambda_i \vec{v}_i$, define

$$C = [\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$$

Then

$$A = C \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{pmatrix} C^{-1}$$

$$= CDC^{-1}$$

We have "diagonalized" A.

(2) 2nd Best: "Jordon Canonical Form".

If $\chi_A(\lambda)$ splits over $\mathbb{F}$ then $\exists$ invertible matrix C such that

$$A = CJC^{-1}$$

$$= C\left(\begin{array}{c|c|c|c} J_1 & 0 & 0 & 0 \\ \hline 0 & J_2 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & J_k \end{array}\right)C^{-1}$$

where $J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$ are "Jordon blocks".

Q: Why do we care?

Because $\forall\, m \geq 0$,

$$A^m = (CJC^{-1})^m$$

$$= CJ\cancel{C^{-1}}\cancel{C}J\cancel{C^{-1}} \cdots \cancel{C}JC^{-1}$$

$$= CJ^mC^{-1}$$

and

$$J^m = \begin{pmatrix} J_1^m & & & \\ & J_2^m & & \\ & & \ddots & \\ & & & J_k^m \end{pmatrix}$$

and (Exercise) if J_i is $b \times b$ then

$$J_i^m = \begin{pmatrix} \lambda_i^m & \binom{m}{1}\lambda_i^{m-1} & \cdots & \binom{m}{b-1}\lambda_i^{m-b+1} \\ & \lambda_i^m & & \vdots \\ & & \ddots & \binom{m}{1}\lambda_i^{m-1} \\ & 0 & & \lambda_i^m \end{pmatrix}$$

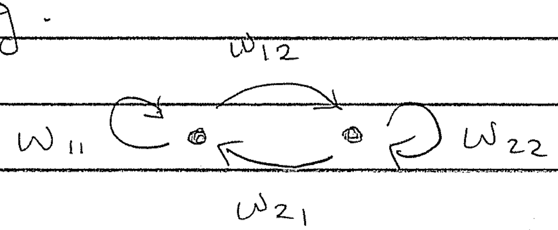
$\longrightarrow 0$ if $\lambda_i \in \mathbb{C}$
$|\lambda_i| < 1$

[In the background: Modules over a PID. See Garrett's notes.]

Q: What is a "matrix"?

Answer 2: A weighted, directed graph.

e.g.



$$= \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

graph G matrix T_G

Observe:

$$T_G^2 = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

$$= \begin{pmatrix} w_{11}w_{11} + w_{12}w_{21} & w_{11}w_{12} + w_{12}w_{22} \\ w_{21}w_{11} + w_{22}w_{21} & w_{21}w_{12} + w_{22}w_{22} \end{pmatrix}$$

What does it mean?

Def: Consider graph G on n vertices with transition matrix $T_G = (w_{ij})_{i,j \in [n]}$

Given a path p in G:

$$p = V_{i_0} \xrightarrow{\omega_{i_0 i_1}} V_{i_1} \xrightarrow{\omega_{i_1 i_2}} \cdots \xrightarrow{\omega_{i_{m-1} i_m}} V_{i_m}$$

define the "weight" of the path

$$wt(p) := \omega_{i_0 i_1} \omega_{i_1 i_2} \cdots \omega_{i_{m-1} i_m}$$

☆ Fundamental Lemma: Let $T_G = (\omega_{ij})$.

$$(T_G^m)_{i,j} = i,j \text{ entry of } T_G^m$$

$$= \sum_{\substack{i \to j \text{ paths } p \\ \text{using } m \text{ steps}}} wt(p) \quad //$$

Proof: Easy induction. ∎

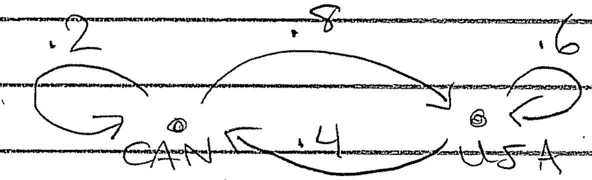
(Think: what if some $\omega_{ij} = 0$?)

☆ Big Idea: Weights can be meaningful!

e.g. If
- $0 \leq \omega_{ij} \leq 1 \quad \forall i,j$
- $\sum_j \omega_{ij} = 1 \quad \forall i$

Then G is called a "Markov Chain"

Example: Butterfly Migration

$G =$ 

$T_G = \begin{pmatrix} .2 & .8 \\ .4 & .6 \end{pmatrix}$

year to year migration probabilities

- $0 \le w_{ij} \le 1$ probabilities
- $\sum_i w_{ij} = 1$ "conservation of butterflies" (rows sum to 1)

$$T_G^2 = \begin{pmatrix} .2\times.2 + .8\times.4 & .8\times.6 + .2\times.8 \\ .6\times.4 + .4\times.2 & .6\times.6 + .4\times.8 \end{pmatrix}$$

$$= \begin{pmatrix} 0.36 & 0.64 \\ 0.32 & 0.68 \end{pmatrix}$$

Corollary to F.L.:

$(T_G^m)_{i,j} = P($ butterfly x is at j now $|$ butterfly x was at i m years ago$)$

$= P(x$ will be at j in m years $|$ x is at i now$)$

Investigate more:

$$T_G^{10} = \begin{pmatrix} 0.33312 & 0.66688 \\ 0.33344 & 0.66656 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix} \ ?$$

Does $\lim_{m\to\infty} T_G^m$ converge?

The answer depends on Jordan Con. Form.

$$\begin{pmatrix} .2 & .8 \\ .4 & .6 \end{pmatrix}^m = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -.2 \end{pmatrix}^m \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-.2)^m \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \end{pmatrix}$$