

Tues, Sept 4

Theme: Convexity in Linear Algebra

Let U, V be vector spaces over field \mathbb{F} .

A morphism $\varphi: U \rightarrow V$ in this category

satisfies: $\forall \vec{x}, \vec{y} \in U, \alpha \in \mathbb{F}$,

$$\bullet \varphi(\vec{x} + \vec{y}) = \varphi(\vec{x}) + \varphi(\vec{y})$$

$$\bullet \varphi(\alpha \vec{x}) = \alpha \varphi(\vec{x})$$

We say φ is an " \mathbb{F} -linear" map.

Given (V, \mathbb{F}) , the set of endomorphisms

$$\text{End}(V, \mathbb{F}) := \{ \mathbb{F}\text{-linear } \varphi: V \rightarrow V \}$$

is itself a vector space. Given

$f, g \in \text{End}(V)$ and $\alpha \in \mathbb{F}$ we define

$\alpha f + g \in \text{End}(V)$ by

$$(\alpha f + g)(\vec{x}) := \alpha f(\vec{x}) + g(\vec{x}) \quad \forall \vec{x} \in V.$$

But End has more structure:

- endomorphisms can be composed!

Def.: An "algebra" is a vector space (V, \mathbb{F}) together with a binary map $\mu: V \times V \rightarrow V$ such that

(1) μ is associative: $\forall \vec{x}, \vec{y}, \vec{z} \in V$,

$$\mu(\vec{x}, \mu(\vec{y}, \vec{z})) = \mu(\mu(\vec{x}, \vec{y}), \vec{z})$$

(2) μ is " \mathbb{F} -bilinear": $\forall \vec{x} \in V$ the maps

$$\mu(\vec{x}, \cdot): V \rightarrow V \quad \& \quad \mu(\cdot, \vec{x}): V \rightarrow V$$

are \mathbb{F} -linear.

[Remarks:

- μ is for "multiplication"
- $(V, +, \mu)$ is a ring.

Prototype: Given v.s. (V, \mathbb{F}) ,

$$(\text{End}(V, \mathbb{F}), +, \circ)$$

↑ composition

is an \mathbb{F} -algebra.

(Note: (End, \circ) is NOT a group.)

The group of units is

$$\begin{aligned} GL(V, \mathbb{F}) &:= \text{End}(V, \mathbb{F})^{\times} \\ &= \{ \text{invertible elements} \} \\ &\text{"general linear group."} \end{aligned}$$

Topic: Coordinates.

Let (V, \mathbb{F}) have $\dim n < \infty$ and choose
a basis, $B = \{ \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n \} \subseteq V$

We encode $\vec{x} \in V$ as a column vector.

If $\vec{x} = a_1 \vec{e}_1 + \dots + a_n \vec{e}_n$ we let

$$[\vec{x}]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$$

Get an isomorphism $[\cdot]_B : (V, \mathbb{F}) \rightarrow \mathbb{F}^n$

Next, we have $\text{End}(V, \mathbb{F}) \leftrightarrow (\mathbb{F}^n)^n$

as sets because $\varphi \in \text{End}(V, \mathbb{F})$ is
determined by the n -tuple of vectors

$$[\varphi(\vec{e}_1)]_B, [\varphi(\vec{e}_2)]_B, \dots, [\varphi(\vec{e}_n)]_B.$$

We encode φ as a "matrix" (Sylvester: "an oblong arrangement of terms")

$$[\varphi]_B := \left([\varphi(\vec{e}_1)]_B \quad \dots \quad [\varphi(\vec{e}_n)]_B \right)$$

\uparrow \uparrow
 columns

Thus if $\vec{v} = a_1 \vec{e}_1 + \dots + a_n \vec{e}_n$ we have

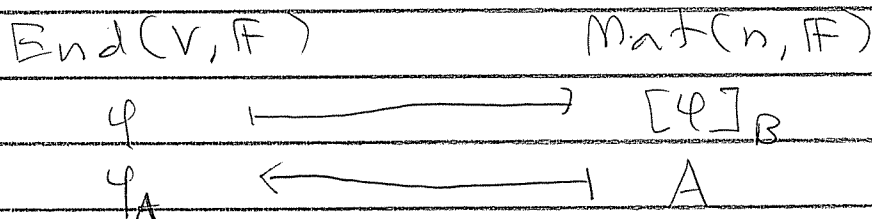
$$\begin{aligned} [\varphi(\vec{v})]_B &= [a_1 \varphi(\vec{e}_1) + \dots + a_n \varphi(\vec{e}_n)]_B \\ &= a_1 [\varphi(\vec{e}_1)]_B + \dots + a_n [\varphi(\vec{e}_n)]_B \end{aligned}$$

$$=: [\varphi]_B [\vec{v}]_B$$

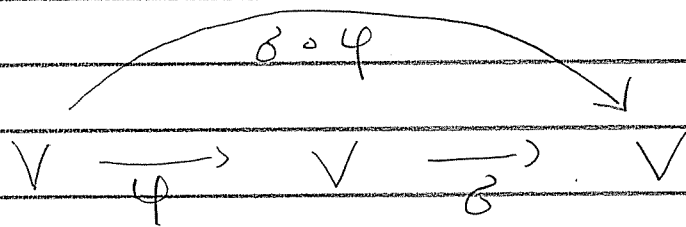
DEFINITION. it is what it is.

Finally, how does composition translate to coordinates?

Def: Let $\text{Mat}(n, \mathbb{F})$ be the set of $n \times n$ matrices. Choosing a basis B for (V, \mathbb{F}) gives inverse bijections



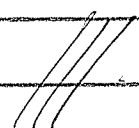
Final Step: Define the matrix product.



If $A = [\delta]_{\beta}$ and $B = [\varphi]_{\beta}$ then

define $AB := [\delta \circ \varphi]_{\beta}$

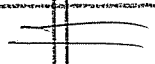
↑ it is what it is.



Summary: Choosing coordinates $\beta \in V$ defines an isomorphism of \mathbb{F} -algebras

$$(\text{End}, +, \circ) \cong (\text{Mat}, +, \text{matrix prod.})$$

$$\varphi \longmapsto [\varphi]_{\beta}$$



Topic: Change of Coordinates:

Given bases $\alpha, \beta \in V \exists$ invertible matrix $C_{\alpha\beta}$ such that

$$[\vec{x}]_{\beta} = C_{\alpha\beta} [\vec{x}]_{\alpha} \quad \& \quad [\vec{x}]_{\alpha} = C_{\alpha\beta}^{-1} [\vec{x}]_{\beta}$$

(If $\alpha = \{\vec{v}_1, \dots, \vec{v}_n\}$, Note that $C_{\alpha\beta} = \begin{bmatrix} [\vec{v}_1]_{\beta} & [\vec{v}_2]_{\beta} & \dots & [\vec{v}_n]_{\beta} \end{bmatrix}$)

Then given $\varphi \in \text{End}$ we have $\forall \vec{x} \in V$,

$$[\varphi(\vec{x})]_{\beta} = C_{\alpha\beta} [\varphi(\vec{x})]_{\alpha}$$

$$[\varphi]_{\beta} [\vec{x}]_{\beta} = C_{\alpha\beta} [\varphi]_{\alpha} [\vec{x}]_{\alpha}$$

$$\underbrace{[\varphi]_{\beta} C_{\alpha\beta}}_{\text{matrix}} [\vec{x}]_{\alpha} = \underbrace{C_{\alpha\beta} [\varphi]_{\alpha}}_{\text{matrix}} [\vec{x}]_{\alpha}$$

Since this holds for all tuples $[\vec{x}]_{\alpha}$
we have a matrix equality (why?)

$$[\varphi]_{\beta} C_{\alpha\beta} = C_{\alpha\beta} [\varphi]_{\alpha}$$

$$[\varphi]_{\beta} = C_{\alpha\beta} [\varphi]_{\alpha} C_{\alpha\beta}^{-1}$$

"similar matrices"

In words: similar matrices encode the same endomorphism in two different bases.

Application: Given $A \in \text{Mat}$ define the characteristic polynomial

$$\chi_A(\lambda) := \det(A - \lambda I)$$

identity matrix

Then if $A = CBC^{-1}$ we have

$$\begin{aligned}\chi_A(\lambda) &= \det(A - \lambda I) \\ &= \det(CBC^{-1} - \lambda CC^{-1}) \\ &= \det(C(B - \lambda I)C^{-1}) \\ &= \det(C) \det(B - \lambda I) \det(C^{-1}) \\ &= \det(B - \lambda I) \\ &= \chi_B(\lambda).\end{aligned}$$

The polynomial depends on the underlying endomorphism, not on the coordinates.

\Rightarrow We can define $\chi_\varphi(\lambda)$ for $\varphi \in \text{End}$.

Q: What if $\chi_\varphi(\lambda) = 0$?

Then $\det(A - \lambda I) = 0$

$$\Rightarrow \ker(A - \lambda I) \neq \{\vec{0}\}$$

$$\Rightarrow \exists \vec{x} \in V \text{ with}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$A\vec{x} = \lambda\vec{x}$$

\uparrow \uparrow
eigenvalue λ -eigenvector.

☆ Big Idea: Given $\varphi \in \text{End}(V, \mathbb{F})$,
choose a basis $\beta \subseteq V$ so $[\varphi]_\beta$ is
as nice as possible

① Best Choice:

If V has a basis $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ of
eigenvectors for φ
(say $\varphi \vec{v}_i = \lambda_i \vec{v}_i$ for some $\lambda_i \in \mathbb{F}$)

Then $[\varphi]_\beta = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$ 😊

(Q: Why are diagonal matrices "nice"?)

Here we say φ is "diagonalizable".

[Fact (Spectral Theorem): IF $A^*A = AA^*$
(say A is "normal") then A is diagonalizable]

But "most" matrices aren't diagonalizable ☹

(2) 2nd Best Choice:

If the char poly $\chi_\varphi(\lambda)$ splits over \mathbb{F}
(e.g. if $\mathbb{F} = \mathbb{C}$). then \exists basis
 $\beta \in V$ such that

$$[\varphi]_\beta = \begin{pmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_k \end{pmatrix}$$

"Jordan canonical form"

where $J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \lambda_i \end{pmatrix}$

a "Jordan block"

Q: Why is this "nice"?