Thur, Sept 4

Theme: Convexity in Linear Algebra

Let U, V be vector spaces over field $\mathbb{F}$.
A morphism $\varphi : U \to V$ in this category satisfies: $\forall \vec{x}, \vec{y} \in U$, $\alpha \in \mathbb{F}$,

- $\varphi(\vec{x}+\vec{y}) = \varphi(\vec{x}) + \varphi(\vec{y})$
- $\varphi(\alpha \vec{x}) = \alpha \varphi(\vec{x})$

We say φ is an "$\mathbb{F}$-linear" map.

Given $(V, \mathbb{F})$, the set of endomorphisms

$$\mathrm{End}(V, \mathbb{F}) := \{ \mathbb{F}\text{-linear } \varphi : V \to V \}$$

is itself a vector space. Given $f, g \in \mathrm{End}(V)$ and $\alpha \in \mathbb{F}$ we define $\alpha f + g \in \mathrm{End}(V)$ by

$$(\alpha f + g)(\vec{x}) := \alpha f(\vec{x}) + g(\vec{x}) \quad \forall \vec{x} \in V.$$

But End has more structure:
- endomorphisms can be composed!

Def: An "algebra" is a vector space $(V, \mathbb{F})$ together with a binary map $\mu : V \times V \to V$ such that

(1) μ is associative: $\forall \vec{x}, \vec{y}, \vec{z} \in V$,

$$\mu(\vec{x}, \mu(\vec{y}, \vec{z})) = \mu(\mu(\vec{x}, \vec{y}), \vec{z})$$

(2) μ is "$\mathbb{F}$-bilinear": $\forall \vec{x} \in V$ the maps

$$\mu(\vec{x}, \circ) : V \to V \quad \& \quad \mu(\circ, \vec{x}) : V \to V$$

are $\mathbb{F}$-linear.

[Remarks:
- μ is for "multiplication"
- $(V, +, \mu)$ is a ring.]

Prototype: Given v.s. $(V, \mathbb{F})$,

$$(\mathrm{End}(V, \mathbb{F}), +, \underbrace{\circ}_{\text{composition}})$$

is an $\mathbb{F}$-algebra.
(Note: $(\mathrm{End}, \circ)$ is NOT a group.

The group of units is

$$GL(V,\mathbb{F}) := End(V,\mathbb{F})^{\times}$$
$$= \{ \text{invertible elements} \}$$

"general linear group".

Topic: Coordinates.

Let $(V,\mathbb{F})$ have $\dim n < \infty$ and <u>choose</u> <u>a basis</u>. $B = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} \subseteq V$

We encode $\vec{x} \in V$ as a column vector. If $\vec{x} = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n$ we let

$$[\vec{x}]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$$

Get an isomorphism $[\cdot]_B : (V,\mathbb{F}) \to \mathbb{F}^n$

Next, we have $End(V,\mathbb{F}) \leftrightarrow (\mathbb{F}^n)^n$ as sets because $\varphi \in End(V,\mathbb{F})$ is determined by the n-tuple of vectors

$$[\varphi(\vec{e}_1)]_B, [\varphi(\vec{e}_2)]_B, \ldots, [\varphi(\vec{e}_n)]_B.$$

We encode φ as a "matrix" (Sylvester: "an oblong arrangement of terms")

$$[\varphi]_B := \Big(\underbrace{[\varphi(\vec{e}_1)]_B \quad \cdots \quad [\varphi(\vec{e}_n)]_B}_{\text{columns}} \Big)$$

Thus if $\vec{v} = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n$ we have

$$\begin{aligned} [\varphi(\vec{v})]_B &= [a_1\varphi(\vec{e}_1) + \cdots + a_n\varphi(\vec{e}_n)]_B \\ &= a_1[\varphi(\vec{e}_1)]_B + \cdots + a_n[\varphi(\vec{e}_n)]_B \\ &=: [\varphi]_B [\vec{v}]_B \end{aligned}$$

DEFINITION. it is what it is.

Finally, how does composition translate to coordinates?

Def: Let $\mathrm{Mat}(n,\mathbb{F})$ be the set of $n\times n$ matrices. Choosing a basis B for $(V,\mathbb{F})$ gives inverse bijections

$$\begin{array}{ccc} \mathrm{End}(V,\mathbb{F}) & & \mathrm{Mat}(n,\mathbb{F}) \\ \varphi & \longmapsto & [\varphi]_B \\ \varphi_A & \longleftarrow\!\shortmid & A \end{array}$$

Final Step: Define the matrix product.

$$\begin{tikzcd} V \arrow[r, "\varphi"'] \arrow[rr, bend left, "\sigma \circ \varphi"] & V \arrow[r, "\sigma"'] & V \end{tikzcd}$$

If $A = [\sigma]_\beta$ and $B = [\varphi]_\beta$ then

define $AB := [\sigma \circ \varphi]_\beta$

it is what it is. ///

Summary: Choosing coordinates $\beta \subseteq V$ defines an isomorphism of $\mathbb{F}$-algebras

$$(\mathrm{End}, +, \circ) \cong (\mathrm{Mat}, +, \text{matrix prod.})$$
$$\varphi \longmapsto [\varphi]_\beta$$

Topic: Change of Coordinates.

Given bases $\alpha, \beta \subseteq V$ $\exists$ invertible matrix $C_{\alpha\beta}$ such that

$$[\vec{x}]_\beta = C_{\alpha\beta}[\vec{x}]_\alpha \quad \& \quad [\vec{x}]_\alpha = C_{\alpha\beta}^{-1}[\vec{x}]_\beta.$$

(If $\alpha = \{\vec{v}_1, \ldots, \vec{v}_n\}$, Note that $C_{\alpha\beta} = \left[[\vec{v}_1]_\beta \; [\vec{v}_2]_\beta \; \cdots \; [\vec{v}_n]_\beta \right]$)

Then given $\varphi \in End$ we have $\forall \vec{x} \in V$,

$$[\varphi(\vec{x})]_\beta = C_{\alpha\beta} [\varphi(\vec{x})]_\alpha$$

$$[\varphi]_\beta [\vec{x}]_\beta = C_{\alpha\beta} [\varphi]_\alpha [\vec{x}]_\alpha$$

$$\underbrace{[\varphi]_\beta C_{\alpha\beta}} [\vec{x}]_\alpha = \underbrace{C_{\alpha\beta} [\varphi]_\alpha} [\vec{x}]_\alpha$$

Since this holds for all tuples $[\vec{x}]_\alpha$ we have a matrix equality (Why?)

$$[\varphi]_\beta C_{\alpha\beta} = C_{\alpha\beta} [\varphi]_\alpha$$

$$[\varphi]_\beta = C_{\alpha\beta} [\varphi]_\alpha C_{\alpha\beta}^{-1}$$

"similar matrices".

In words: similar matrices encode the <u>same</u> endomorphism in two <u>different</u> bases.

Application: Given $A \in Mat$ define the characteristic polynomial

identity matrix

$$\chi_A(\lambda) := \det(A - \lambda I)$$

Then if $A = CBC^{-1}$ we have

$$\begin{aligned}
\chi_A(\lambda) &= \det(A - \lambda I) \\
&= \det(CBC^{-1} - \lambda CC^{-1}) \\
&= \det(C(B - \lambda I)C^{-1}) \\
&= \cancel{\det}(C) \det(B - \lambda I) \cancel{\det}(C^{-1}) \\
&= \det(B - \lambda I) \\
&= \chi_B(\lambda).
\end{aligned}$$

The polynomial depends on the underlying endomorphism, <u>not</u> on the coordinates.

$\Rightarrow$ We can define $\chi_\varphi(\lambda)$ for $\varphi \in \mathrm{End}$.

Q: What if $\chi_\varphi(\lambda) = 0$?

Then $\det(A - \lambda I) = 0$

$\Rightarrow \ker(A - \lambda I) \neq \{\vec{0}\}$

$\Rightarrow \exists\, \vec{x} \in V$ with

$$(A - \lambda I)\vec{x} = \vec{0}$$
$$A\vec{x} - \lambda\vec{x} = \vec{0}$$
$$A\vec{x} = \lambda\,\vec{x}$$

eigenvalue $\quad$ λ-eigenvector.

☆ Big Idea: Given $\varphi \in End(V, \mathbb{F})$, choose a basis $\beta \subseteq V$ so $[\varphi]_\beta$ is <u>as nice as possible</u>

(1) Best Choice:
If V has a basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ of eigenvectors for φ
(say $\varphi \vec{v}_i = \lambda_i \vec{v}_i$ for some $\lambda_i \in \mathbb{F}$)

$$\text{Then } [\varphi]_\beta = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$:)

(Q: Why are diagonal matrices "nice"?)

Here we say φ is "diagonalizable.

[Fact (Spectral Theorem): If $A^*A = AA^*$ (say A is "normal") then A is diagonalizable.]

But "most" matrices aren't diagon'ble :(

(2) 2nd Best Choice:

If the char poly $\chi_\varphi(\lambda)$ splits over $\mathbb{F}$ (e.g. if $\mathbb{F} = \mathbb{C}$). then $\exists$ basis $\beta \in V$ such that

$$[\varphi]_\beta = \left(\begin{array}{cc|c|c} J_1 & 0 & 0 & 0 \\ \hline 0 & J_2 & 0 & 0 \\ \hline 0 & 0 & \ddots & \\ \hline 0 & 0 & 0 & J_k \end{array}\right)$$

"Jordan canonical form"

where $J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$

a "Jordan block"

Q: Why is this "nice"?