Tues Nov 27
(2012)

Recall :

☆ Cartan-Dieudonné Theorem ☆

Let $Q : V \to \mathbb{F}$ be non-degenerate with $\dim V = n$ and $\operatorname{char} \mathbb{F} \neq 2$. Then for all $\varphi \in O(V,Q)$, $\exists$ anisotropic $u_1, u_2, \ldots, u_k \in V$ with $k \leq n$ such that

$$\varphi = R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_k}$$

"The group $O(V,Q)$ is generated by reflections"

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Today we'll extend this result to the full isometry group

$$\mathrm{Isom}(AV, Q) = V \rtimes O(V,Q).$$

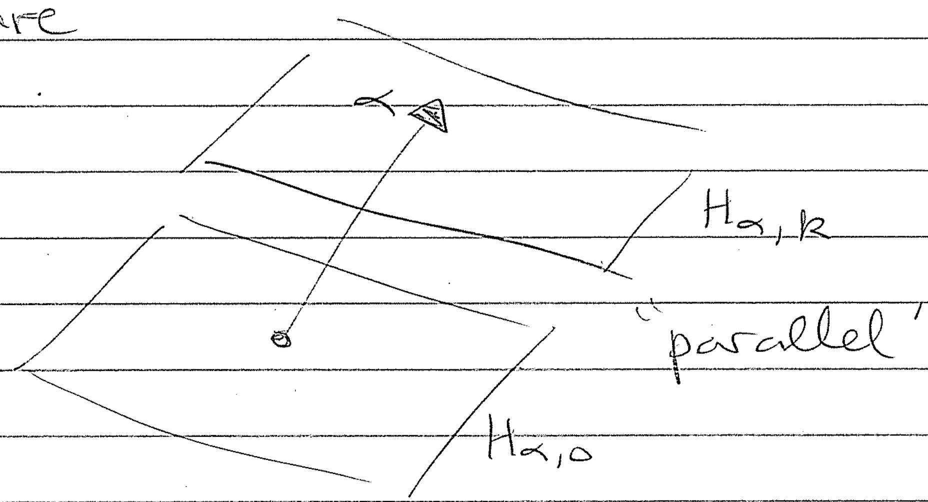
to show : every $\varphi \in \mathrm{Isom}(AV, Q)$ is a product of at most $\underline{n+1}$ "affine reflections".

Let $B : V \times V \to \mathbb{F}$ be symmetric and consider some $\alpha \in V$ with $B(\alpha, \alpha) \neq 0$. For all $k \in \mathbb{F}$ we define the "affine hyperplane"

$$H_{\alpha,k} := \{ x \in V : B(x, \alpha) = k \}$$

(Note $H_{\alpha,0} = \alpha^{\perp}$)

Picture



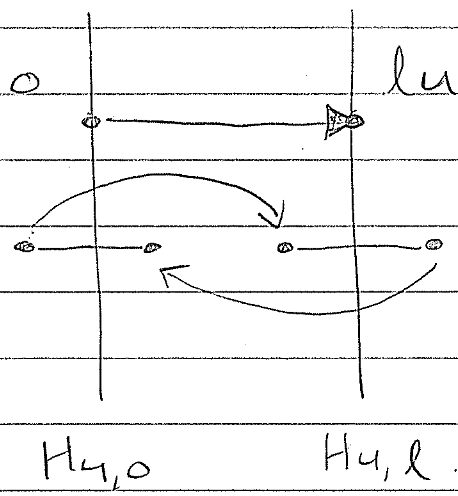
For all $\lambda \neq 0$ we have

$$H_{\alpha,k} = H_{\lambda\alpha, \lambda k}$$

so you can assume $B(\alpha, \alpha) = 1$ ("unit vector") if you want. That is, if $\sqrt{B(\alpha,\alpha)} \in \mathbb{F}$ ...

For simplicity, assume $\sqrt{B(\alpha,\alpha)} \in \mathbb{F}$ and let $u = \alpha / \sqrt{B(\alpha,\alpha)}$, so $B(u,u) = 1$.

Then $B(lu, u) = \quad \Longrightarrow \quad lu \in H_{u,l}$.



Consider the reflection across $H_{u,k}$

$$R_{u,l} := t_{lu} \circ R_{u,0} \circ t_{-lu}.$$

We can compute a formula: $\forall x \in V$,

$$R_{u,l}(x) = t_{lu}(R_u(x - lu))$$

$$= t_{lu}\Big[(x - lu) - 2B(x - lu, u)u\Big]$$

$\wr$

$$= t_{lu}\left[x - lu - 2B(x,u)u + 2l\,\overset{1}{\cancel{B(u,u)}}\,u\right]$$

$$= t_{lu}\left[x - 2B(x,u)u + lu\right]$$

$$= x - 2B(x,u)u + 2lu. \qquad (*)$$

Now recall $u = \alpha / \sqrt{B(\alpha,\alpha)}$. If $l = k/\sqrt{B(\alpha,\alpha)}$ then we have

$$H_{u,l} = H_{\alpha,k}$$

$$\Longrightarrow R_{u,l} = R_{\alpha,k}.$$

Hence by $(*)$ we have

$$R_{\alpha,k}(x) = x - 2B(x,u)u + 2lu.$$

$$= x - \frac{2B(x,\alpha)}{B(\alpha,\alpha)}\alpha + \frac{2k}{B(\alpha,\alpha)}$$

$$\Longrightarrow \boxed{R_{\alpha,k}(x) = x - 2\left(\frac{B(x,\alpha) - k}{B(\alpha,\alpha)}\right)\alpha}$$

True even if $\sqrt{B(\alpha,\alpha)} \notin \mathbb{F}$.

Check : $R_{\alpha,0} = R_\alpha$ ✓

We can lean something else from (*)

$$R_{u,\ell}(x) = \underbrace{x - 2B(x,u)u} + 2\ell u.$$

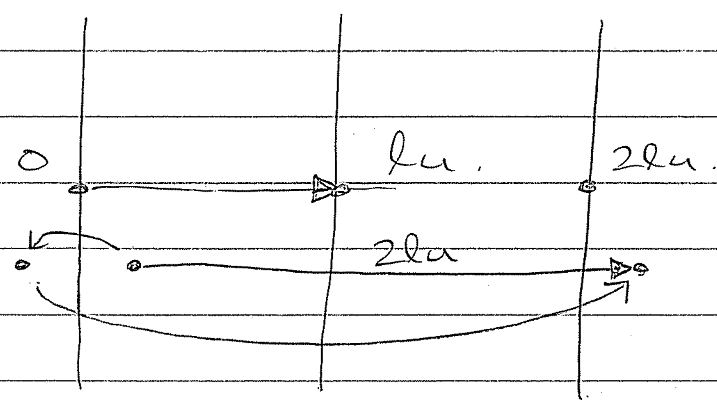
$$= R_{u,0}(x) + 2\ell u.$$

$$= t_{2\ell u} \circ R_{u,0}(x).$$

$$\Longrightarrow \quad R_{u,\ell} = t_{2\ell u} \circ R_{u,0}$$

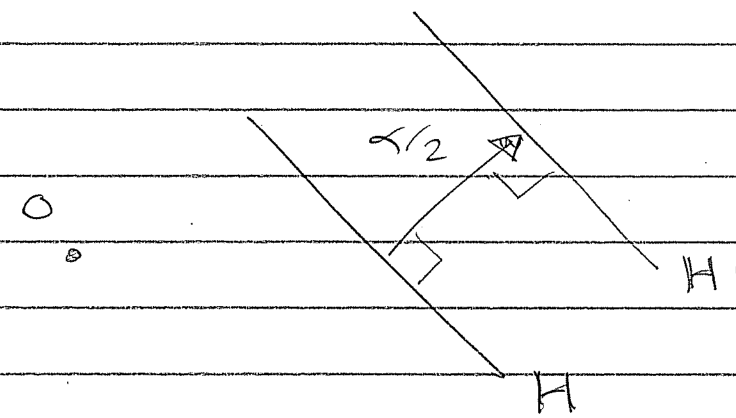
$$\Longrightarrow \quad \boxed{t_{2\ell u} = R_{u,\ell} \circ R_{u,0}}$$

"Translation = Two Parallel Reflections"



$H_{u,0}$ $H_{u,\ell}.$

More generally, let $H, H' \subseteq V$ be hyperplanes with $H' = H + \alpha/2$.



Then $t_\alpha = R_{H'} \circ R_H$

We are now ready to state

Theorem (Affine Cartan-Dieudonné):

Let $Q : V \to \mathbb{F}$ be non-deg. with $\dim V = n$ and $\operatorname{char} \mathbb{F} \neq 2$. Then $\forall$ isometries $\varphi \in \underline{\operatorname{Isom}(AV, Q)}$ $\exists$ anisotropic vectors $u_1, u_2, \ldots, u_m \in V$ with $m \leq n+1$ and a scalar $k \in \mathbb{F}$ such that

$$\varphi = \underset{\text{affine}}{R_{u_1,k}} \circ \underbrace{R_{u_2,0} \circ \cdots \circ R_{u_m,0}}_{\text{linear}}$$

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We need one lemma.

<u>Lemma</u>: Suppose $\varphi \in O(V,Q)$ is the product of $n = \dim V$ reflections. Then we can choose the <u>first</u> reflection <u>arbitrarily</u>.

<u>Proof</u>: Suppose $\varphi = R_1 R_2 \cdots R_n$ and let R be <u>any</u> reflection. Then $R\varphi \in O(V,Q)$ so C-D $\Rightarrow \exists$ some reflections $R_1', \ldots, R_k'$ such that

$$R\varphi = R_1' R_2' \cdots R_k' \quad (k \leq n).$$

$$\Rightarrow \varphi = R R_1' R_2' \cdots R_k'$$

Since $\det(\varphi) = (-1)^n = (-1)^{k+1}$ and since $k \leq n$ we conclude that $k \leq n-1$.

$\blacksquare$

Proof of Affine C-D :

Let $\varphi \in \mathrm{Isom}(AV, Q)$. Since $\mathrm{Isom}(AV, Q) = V \rtimes O(V, Q)$ we have

$$\varphi = t_\alpha \circ A$$

for unique $\alpha \in V$ and $A \in O(V, Q)$.
By Linear C-D. $\exists$ linear reflections $R_1, R_2, \ldots, R_k$ with $k \leq n$ and

$$A = R_1 \circ R_2 \circ \cdots \circ R_k$$

There are 2 cases.

Case 1 : If $k \leq n-1$, then write $t_\alpha = R_{\alpha, \frac{B(\alpha,\alpha)}{2}} \circ R_{\alpha, 0}$, hence

$$\varphi = \underbrace{R_{\alpha, \frac{B(\alpha,\alpha)}{2}} \circ R_{\alpha, 0} \circ R_1 \circ R_2 \circ \cdots \circ R_k}_{\leq n+1 \text{ reflections}}$$

Case 2: If $k = n$, use the Lemma to write

$$A = R_{\alpha,0} \circ R_1' \circ R_2' \circ \cdots \circ R_{n-1}'$$

Then we have

$$\varphi = t_\alpha \circ A.$$

$$= R_{\alpha, \frac{B(\alpha,\alpha)}{2}} \circ \underbrace{\cancel{R_{\alpha,0} \circ R_{\alpha,0}}}_{id.} \circ R_1' \circ \cdots \circ R_{n-1}'$$

$$= \underbrace{R_{\alpha, \frac{B(\alpha,\alpha)}{2}} \circ R_1' \circ \cdots \circ R_{n-1}'}_{\leq n \text{ reflections}}$$

$\blacksquare$