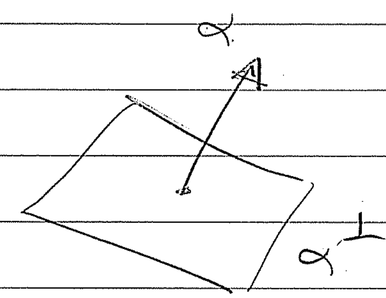
Tues Nov 20

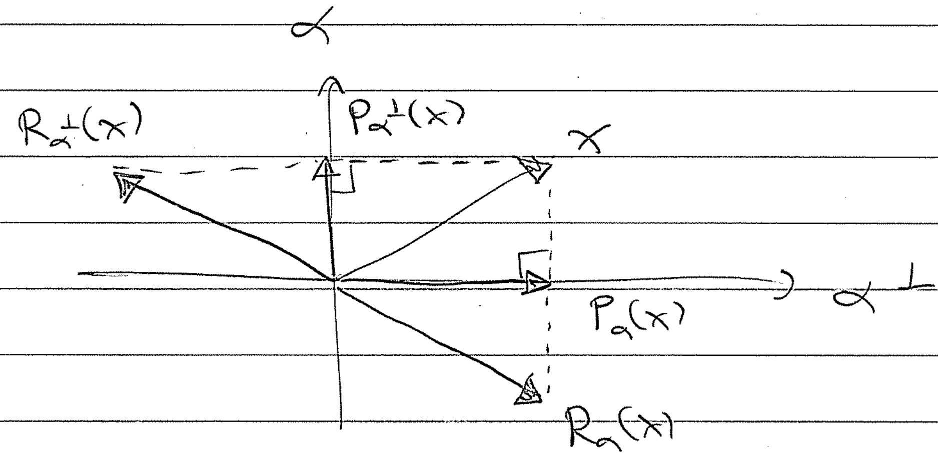
Let $Q : V \to \mathbb{F}$ be a quadratic form.

Recall: Given any "anisotropic" vector $Q(\alpha) \neq 0$ we get an orthogonal direct sum

$$V = \underbrace{\alpha}_{\text{line}} \perp \underbrace{\alpha^{\perp}}_{\text{hyperplane}}$$



Then we can define orthogonal projections & reflections



Related by

① $P_\alpha + P_{\alpha^\perp} = I$ (complete)

② $P_\alpha P_{\alpha^\perp} = P_{\alpha^\perp} P_\alpha = 0$ (orthogonal)

③ $P_\alpha^2 = P_\alpha$ (idempotents)
$P_{\alpha^\perp}^2 = P_{\alpha^\perp}$

and (4) $R_\alpha = 2P_\alpha - I$

$R_{\alpha^\perp} = 2P_{\alpha^\perp} - I$.

It follows that $R_\alpha + R_{\alpha^\perp} = 0$

i.e. $R_\alpha(x) = -R_{\alpha^\perp}(x) \quad \forall\, x \in V$.

Let $B(x,y) = \frac{1}{2}\left[Q(x+y) - Q(x) - Q(y) \right]$

Then we can write

$$\boxed{R_\alpha(x) = x - \frac{2B(x,\alpha)}{B(\alpha,\alpha)}\alpha}$$

Reflection across $\alpha^\perp$

Note: R_α preserves Q.

$\forall\, x, y \in V$ we have

$$B(R_\alpha(x), R_\alpha(y))$$

$$= B\left(x - \frac{2B(x,\alpha)}{B(\alpha,\alpha)}\alpha,\ y - \frac{2B(y,\alpha)}{B(\alpha,\alpha)}\alpha\right)$$

$$= B(x,y) - \frac{2B(x,\alpha)}{B(\alpha,\alpha)}B(\alpha,y)$$

$$- \frac{2B(y,\alpha)}{B(\alpha,\alpha)}B(x,\alpha) + \frac{4B(x,\alpha)B(y,\alpha)B(\alpha,\alpha)}{B(\alpha,\alpha)^2}$$

$= B(x,y)$

$\Longrightarrow R_\alpha \in O(V,Q)$
"isometry"

In coordinates we have

$$R_\alpha = \begin{array}{cc} \left(\begin{array}{c|ccc} -1 & & & \\ \hline & 1 & & \\ & & \ddots & \\ & & & 1 \end{array}\right) & \begin{array}{c} \alpha \\ \alpha^\perp \end{array} \\ \begin{array}{cc} \alpha & \qquad \alpha^\perp \end{array} & \end{array}$$

Hence $\det R_\alpha = -1$ (i.e. $R_\alpha \in O^-(V)$)

And that's all:

Theorem (Cartan <1948, Dieudonné 1948):

Let $Q : V \to \mathbb{F}$ be non-degenerate and $\dim V = n$ with $\operatorname{char} \mathbb{F} \neq 2$. Then $\forall\, \varphi \in O(V,Q)$, $\exists$ anisotropic vectors $u_1, u_2, \ldots, u_k \in V$ $(k \leq n)$ such that

$$\varphi = R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_k}$$

Proof: Let $\varphi \in O(V, Q)$. We will prove that $\varphi = R_{u_1} \circ \cdots \circ R_{u_k}$ $(k \leq n)$ by induction on n.

(anisotropic)

Since Q is non-deg. $\exists$ $Q(x) \neq 0$.

Proof: $\exists$ u, w with $B(u,w) \neq 0$. If $B(u,u) \neq 0$ or $B(w,w) \neq 0$, done. Otherwise let $x = u + w$. Then

$$
\begin{aligned}
B(x,x) &= B(u+w, u+w) \\
&= \underbrace{B(u,u)}_{0} + 2B(u,w) + \underbrace{B(w,w)}_{0} \\
&= 2B(u,w) \neq 0.
\end{aligned}
$$

///

Case 1: suppose $\exists$ $Q(x) \neq 0$ with $\varphi(x) = x$. Then φ stabilizes the hyperplane $x^\perp$ since if $h \in x^\perp$,

$$0 = B(h, x) = B(\varphi(h), \varphi(x)) = B(\varphi(h), x).$$

By induction, $\varphi|_{x^\perp}$ is a product of $\leq n-1$ reflections

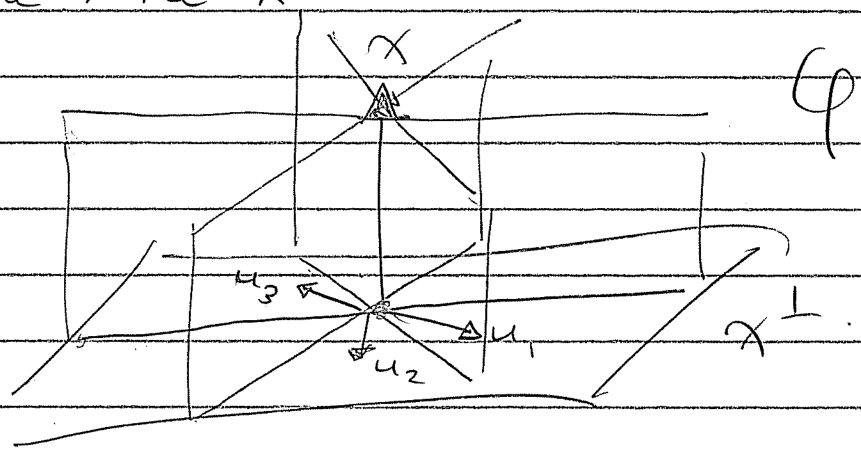
$$\varphi|_{x^\perp} = R_{u_1}|_{x^\perp} \circ \cdots \circ R_{u_k}|_{x^\perp}$$

for some $u_1, u_2, \ldots, u_k \in x^\perp$, $k \leq n-1$.

But then we have

$$\varphi = R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_k} \quad (k \le n-1).$$

since everything acts as the identity on the line x.



Case 2: suppose $\exists\, Q(x) \neq 0$ such that $\varphi(x) \neq x$ but $Q(\varphi(x) - x) \neq 0$.

Note that

$$\begin{aligned}
0 &\neq B(\varphi(x) - x, \varphi(x) - x) \\
&= B(\varphi(x), \varphi(x)) - 2B(\varphi(x), x) + B(x, x) \\
&= 2B(\varphi(x), \varphi(x)) - 2B(\varphi(x), x). \\
&= 2B(\varphi(x), \varphi(x) - x)
\end{aligned}$$

$\Downarrow$

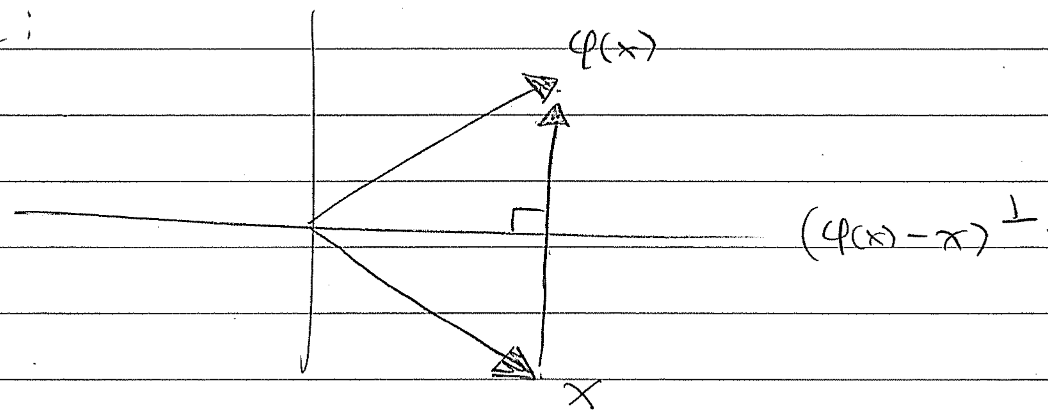
Hence we can reflect:

$$R_{\varphi(x)-x} \circ \varphi(x)$$

$$= \varphi(x) - \frac{2\cancel{B(\varphi(x), \varphi(x)-x)}}{\cancel{B(\varphi(x)-x, \varphi(x)-x)}}^{1} (\varphi(x)-x)$$

$$= \varphi(x) - (\varphi(x) - x) = x.$$

Picture:



Hence $R_{\varphi(x)-x} \circ \varphi \in O(V, Q)$ fixes x and by Case 1 we have

$$R_{\varphi(x)-x} \circ \varphi = R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_k} \quad (k \leq n-1)$$

$$\varphi = \underbrace{R_{\varphi(x)-x} \circ R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_k}}_{\leq n \text{ reflections.}}$$

Case 3: Otherwise we have $\forall\, Q(x) \neq 0$ that $\varphi(x) \neq x$ and $Q(\varphi(x) - x) = 0$.
Uh Oh!

[Technical Lemma: Then we have $n \geq 4$, n is even, and $\varphi \in O^+(V)$.
Proof Omitted (see Pete L. Clark).]

Now let $R \in O^-(V)$ be any reflection, so $R \circ \varphi \in O^-(V)$. By Cases 1 and 2 we have

$$R \circ \varphi = R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_k} \quad (k \leq n)$$

$$\varphi = \underbrace{R \circ R_{u_1} \circ \cdots \circ R_{u_k}}_{\leq n+1 \text{ reflections}}$$

But since n is even, $k = n$
$\Longrightarrow \varphi \in O^-(V)$, contradiction.
Hence $k < n$

$\blacksquare$

Note: Technical Lemma is unnecessary for a totally anisotropic form (like "dot product")

Now let $\varphi = R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_k}$ for some $u_1, u_2, \ldots, u_k \in V$ and consider the <u>Fixed space</u>

$$\begin{aligned} \mathrm{Fix}(\varphi) &:= \ker(\varphi - I) \\ &= \{ x \in V : \varphi(x) - x = 0 \} \\ &= \{ x \in V : \varphi(x) = x \}. \end{aligned}$$

Note that $\mathrm{Fix}(R_\alpha) = \alpha^\perp$, hence

$$\mathrm{Fix}(\varphi) \geq \underbrace{u_1^\perp \cap u_2^\perp \cap \cdots \cap u_k^\perp}_{\text{intersection of hyperplanes}}$$

Note also that

$$\dim(u_1^\perp \cap u_2^\perp \cap \cdots \cap u_k^\perp) \geq n - k$$

with equality $\iff$ $u_1, \ldots, u_k$ are linearly independent.

Hence

$$\dim \mathrm{Fix}(\varphi) \geq n - k.$$

Define the "reflection length" function

$$\ell : O(V,Q) \longrightarrow \mathbb{N}$$

$$\ell(\varphi) := \min\{k : \exists\, u_1, u_2, \dots, u_k \in V \text{ with } \varphi = R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_k}\}$$

$(\ell(I) = 0$ by convention$)$.

Cartan-Dieudonné says:
① $\ell(\varphi)$ exists
② $\ell(\varphi) \leq \dim V \quad \forall \varphi$.

By above remarks, if $\ell(\varphi) = k$ then

$$\dim \mathrm{Fix}(\varphi) \geq n - \ell(\varphi)$$
$$\ell(\varphi) \geq n - \dim \mathrm{Fix}(\varphi)$$
$$\ell(\varphi) \geq \operatorname{codim} \mathrm{Fix}(\varphi)$$

In fact we have
Theorem (Scherk 1950)

For $\varphi \in O(V,B)$ with B symm. non-deg. and char $\mathbb{F} \neq 2$ we have

$$\ell(\varphi) = \operatorname{codim} \operatorname{Fix}(\varphi)$$

unless $\left[B(\varphi - I)\right]^t = -B(\varphi - I)$
(which is very <u>rare</u>), in which case

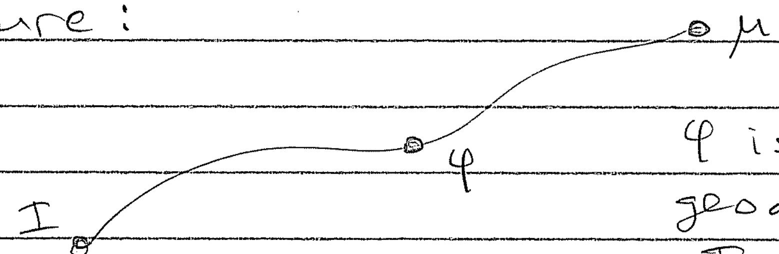
$$\ell(\varphi) = \operatorname{codim} \operatorname{Fix}(\varphi) + 2$$

Modern History (Brady-Watt 2002)

Study the "reflection" partial order on $O(V)$

$$\varphi \le \mu \iff \ell(\mu) = \ell(\varphi) + \ell(\varphi^{-1}\mu).$$

Picture:

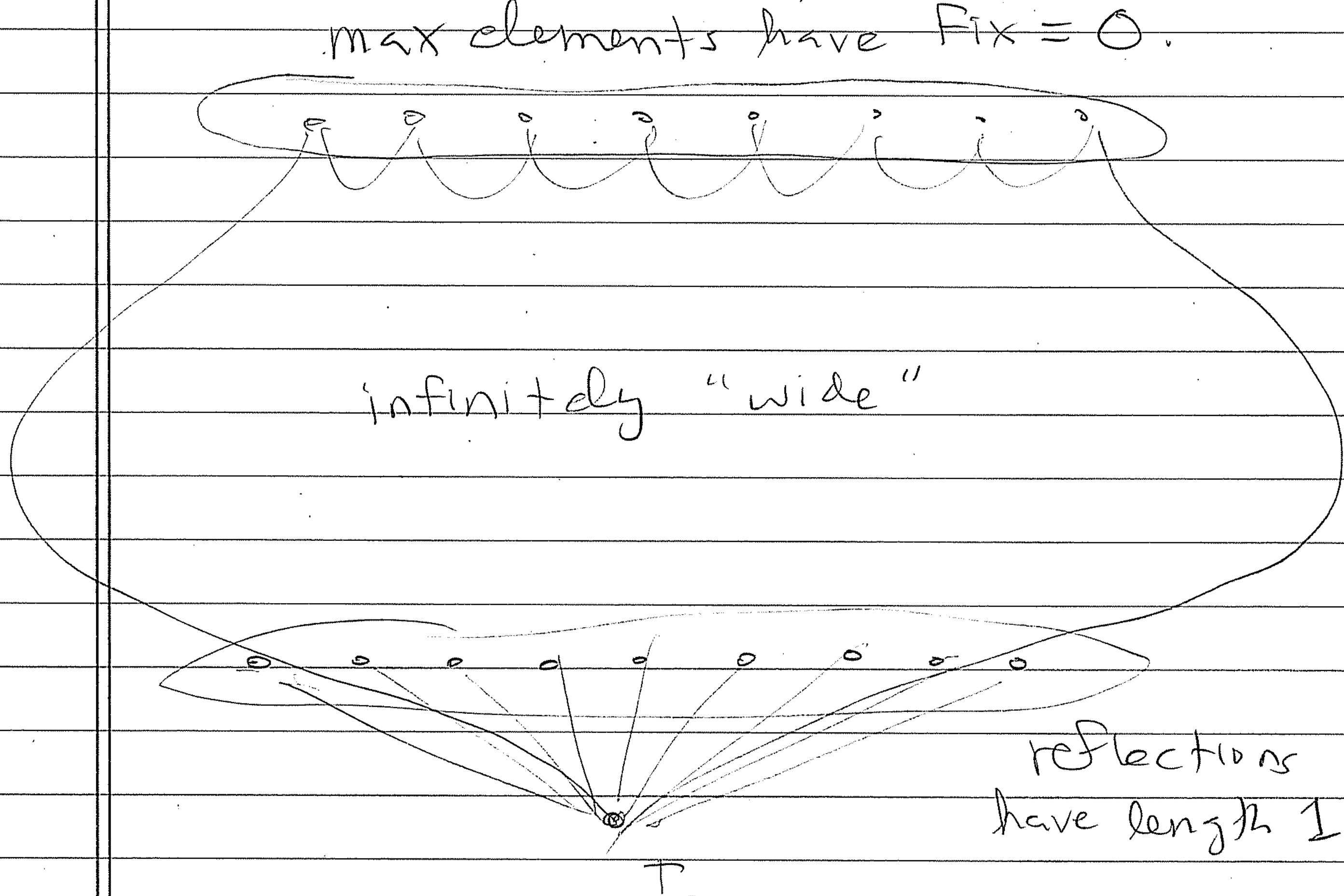


φ is on a geodesic between I and μ.

i.e. $\exists\ \mu = R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_k}$

with <u>prefix</u> $\varphi = R_{u_1} \circ R_{u_2} \circ \cdots \circ R_{u_j}$ $(j \le k)$.

Hasse Diagram of $(O(V,Q), \leq)$ is the Cayley graph of $O(V,Q)$ with respect to the generating set of reflections



Open Problem :

Study $(O(V), \leq)$ over finite fields.