Thurs Nov 8

Let G be a group.
Recall that a "G-torsor" is a set S such that G acts $G \curvearrowright S$ regularly (and preserves some structure) i.e.

(i) Transitive : $\forall x, y \in S \ \exists g \in G, \ g(x) = y$.

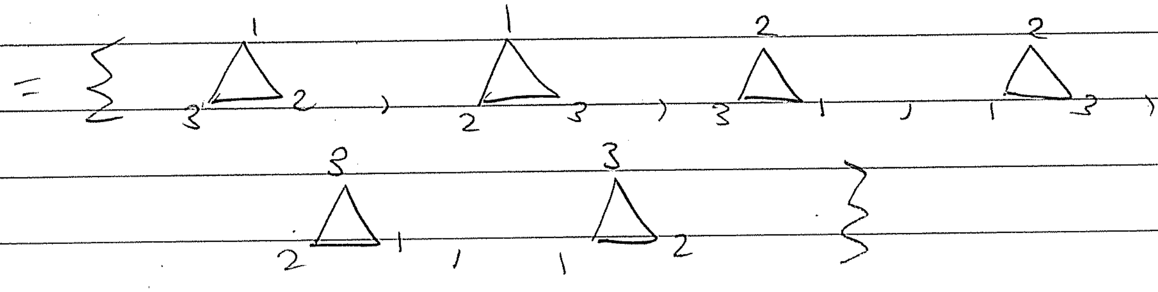
(ii) Free : $\forall x \in S, \ g, h \in G, \ g(x) = h(x) \Rightarrow g = h$.

Then for any choice of "basepoint" $x_0 \in S$ we get a bijection

$$\begin{array}{ccccc} S & \longleftrightarrow & G/G_x & \longleftrightarrow & G \\ g(x) & \longleftrightarrow & gG_x & \longleftrightarrow & g. \end{array}$$

Think: S is a group ($\approx G$) with "identity element" $x_0 \in S$. But the choice of x_0 is arbitrary (i.e. NON-CANONICAL).

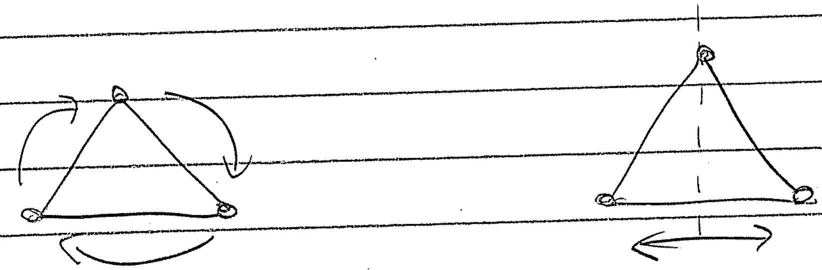
Example: Let $S = \{$labeled triangles$\}$



The dihedral group D_6 is generated by ρ and φ such that $\rho^3 = 1$, $\varphi^2 = 1$ and $\varphi \rho \varphi^{-1} = \rho^{-1}$.

$$D_6 = \langle \varphi, \rho : \varphi^2 = \rho^3 = \varphi\rho\varphi\rho = 1 \rangle$$

Then $D_6 \curvearrowright S$ regularly, where



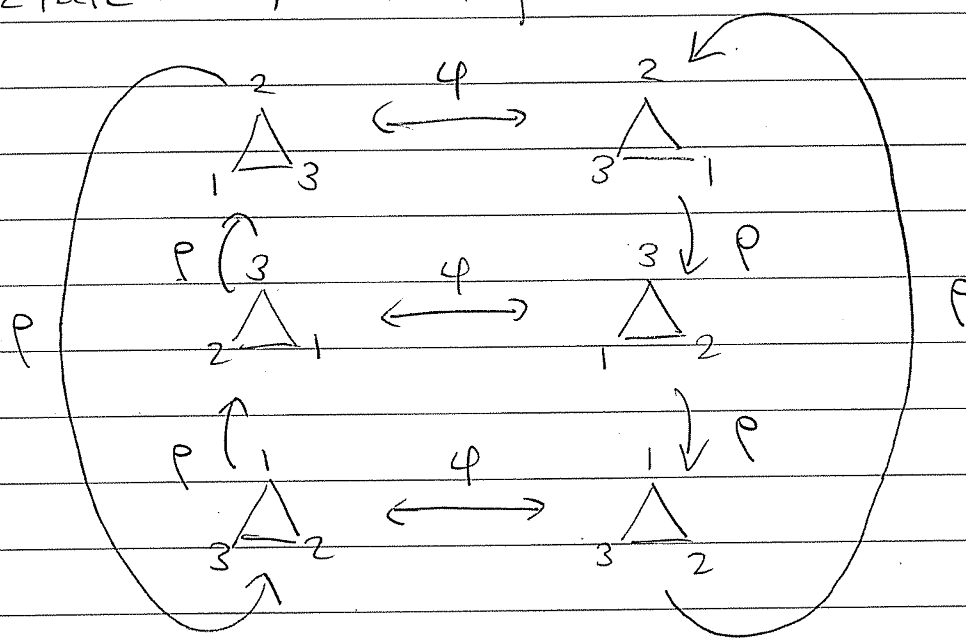
ρ = "rotation" $\qquad$ φ = "flip"

Put a "directed graph" structure on S by saying

$$x \to y \iff \rho(x) = y \text{ OR } \varphi(x) = y.$$

Then $(S, \to)$ is a D_6-torsor, called the "Cayley graph" of D_6 with respect to generating set $\{\rho, \varphi\}$.

Picture: "twisted prism"



You can "see" that $D_6 = \langle \rho \rangle \rtimes \langle \varphi \rangle$

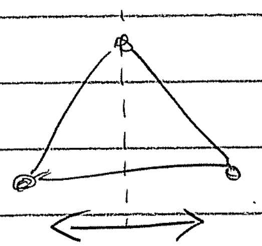
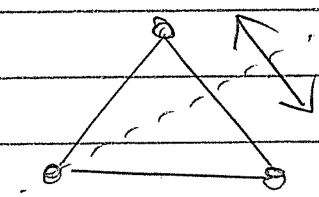
Now choose an <u>arbitrary</u> basepoint:

$$
\begin{tikzcd}
\rho\varphi \arrow[r, leftrightarrow, "\varphi"] \arrow[dd, bend right=60, "\rho"'] & \rho^2 \arrow[d, "\rho"] \\
\varphi \arrow[u, "\rho"] \arrow[r, leftrightarrow, "\varphi"] & 1 \arrow[d, "\rho"] \\
\rho^2\varphi \arrow[u, "\rho"] \arrow[r, leftrightarrow, "\varphi"] & \rho \arrow[uu, bend right=60, "\rho"']
\end{tikzcd}
$$

Example: D_6 can also be presented as

$$D_6 = \langle s, t : s^2 = t^2 = ststst = 1 \rangle$$

Then $D_6 \curvearrowright S$ regularly, where

s = reflection $\qquad\qquad$ t = reflection.

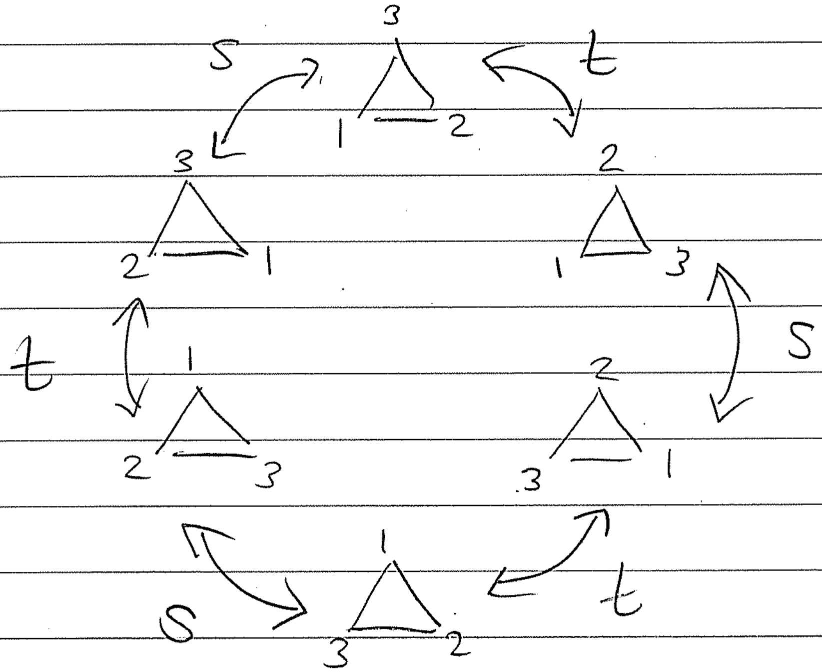
(D_6 is generated by reflections)

Consider the "Cayley graph" of D_6 with respect to generators $\{s, t\}$.
i.e. $\forall\, x, y \in S$ put

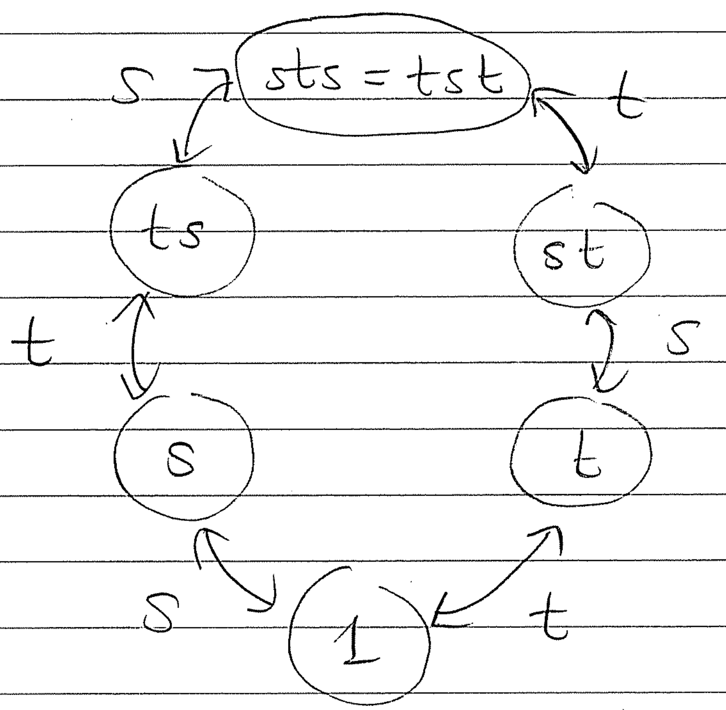
$$x \to y \iff s(x) = y \text{ OR } t(x) = y.$$

Again, $(S, \to)$ is a D_6-torsor.

Picture: "hexagon"



Now choose an arbitrary basepoint.

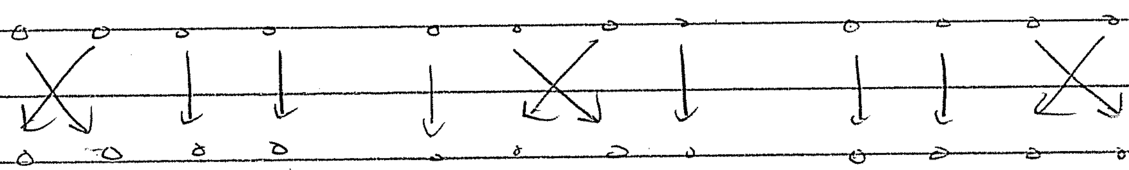


Example: The symmetric group S_4 is generated by a, b, c where

$$acac = 1$$

$$S_4 = \langle a, b, c : a^2 = b^2 = c^2 = ababab = bcbcbc = 1 \rangle$$

Then $S_4 \curvearrowright \{\text{permutations of } 1,2,3,4\}$ regularly by



$a \qquad\qquad b \qquad\qquad c$

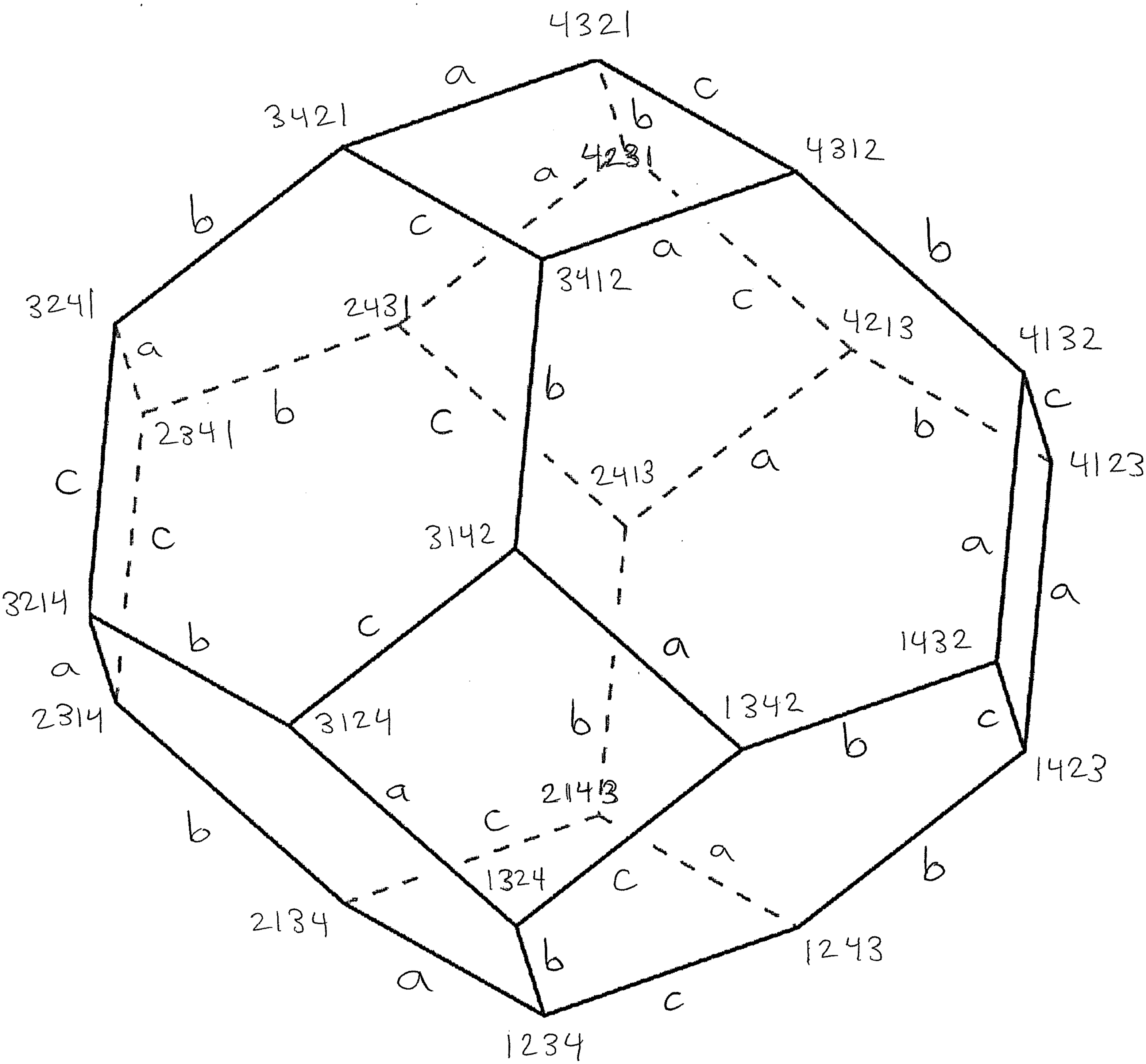
(called "adjacent transpositions")

Note: $\langle a,b\rangle \approx D_6$, $\langle b,c\rangle \approx D_6$

The a, b, c - Cayley graph is an S_4 - torsor, called the

"permutahedron"

(see handout).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