Thur Oct 30.

Some business:

HW 1 due next week.

Class maybe cancelled Thurs.

Sign up for second semester please.

Today: "Affine" geometry.

Let $\mathbb{F}$ be a field and consider $A \in \mathrm{Mat}_n(\mathbb{F})$ and $b \in \mathbb{F}^n$. We know that

$$V = \{ x \in \mathbb{F}^n : Ax = 0 \}$$

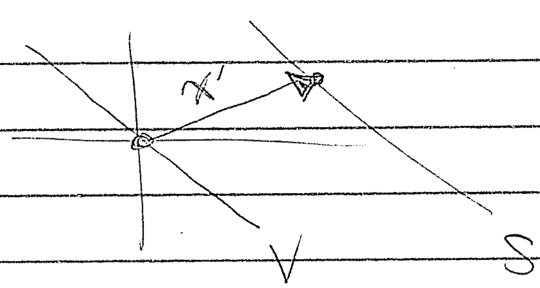
is a vector space over $\mathbb{F}$.

Q: What is

$$S = \{ x \in \mathbb{F}^n : Ax = b \} \; ?$$

Answer 1:

S is a coset of V inside $\mathbb{F}^n$. (as abelian groups).



Given any one particular solution (coset representative) $Ax' = b$, we have

$$S = V + x'$$

Answer 2: For any two $x, y \in S$ we have

$$A(x-y) = Ax - Ay = b - b = 0$$

$$\Rightarrow x - y \in V$$

Think: V is the space of "directions" in S. S is like a vector space but it doesn't have a distinguished <u>origin</u>.

Say S is an "affine space" associated to V (we forget the origin)

<u>Axiomatically</u>: Let V be a vector space. An affine space is a set S with a function $\Theta : S \times S \to V$ such that

1. $\forall x \in S$, $\Theta(x, \cdot) : S \to V$ is surjective
2. $\forall x, y \in S$, $\Theta(x,y) = 0 \Rightarrow x = y$.
3. $\forall x, y, z \in S$ we have
$$\Theta(x,y) + \Theta(y,z) = \Theta(x,z)$$

///

Think: $\Theta(x,y) = \overrightarrow{xy} =$ "$x-y$"
a "displacement vector"

Note: Actually $\Theta(x,\circ): S \to V$ is a <u>bijection</u> since if $\Theta(x,y) = \Theta(x,z)$ then

$$\Theta(x,y) + \Theta(y,z) = \Theta(x,z) = \Theta(x,y)$$

$$\implies \Theta(y,z) = 0 \implies y = z \qquad \boxtimes$$

thus any point $x \in S$ defines an isomorphism

$$\Theta(x,\circ): S \xrightarrow{\sim} V$$

called "choosing an origin".
The choice is not canonical, and that's the whole point.

Commonly we just write

$$S = \mathbb{A}V \quad \text{"affine } V\text{"}$$

$$S = \mathbb{A}\mathbb{F}^n \quad \text{in coordinates.}$$

Thurs Nov 3

"Affine" Geometry.

DEF: Let G = abstract group with subgroups $H, K \leq G$. If

- $G = HK := \{hk : h \in H, k \in K\}$
- $H \cap K = 1$

then every $g \in G$ can be written as $g = hk$ for some $h \in H$, $k \in K$ and the expression is unique:

$$g = h_1 k_1 = h_2 k_2$$
$$h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K.$$
$$\Longrightarrow h_2^{-1} h_1 = k_2 k_1^{-1} = 1.$$
$$\Longrightarrow h_1 = h_2 \ \& \ k_1 = k_2 \qquad ///$$

We get a bijection

$$G \longleftrightarrow H \times K = \{(h,k) : h \in H, k \in K\}$$

as sets.

Q: Group structure?

If $K \trianglelefteq G$ (normal) then $H \curvearrowright K$ by conjugation $\forall h \in H, k \in K$ $k \mapsto h^{-1} k h \in K$.
Group product on G is

$$(h_1 k_1)(h_2 k_2) = \underbrace{h_1 h_2}_{\in H} \underbrace{(h_2^{-1} k_1 h_2) k_2}_{\in K}$$

Notation: $G = H \ltimes K$
"semi-direct product"

Abstractly: Given groups H, K and group hom $\varphi : H \to \mathrm{Aut}(K)$
$h \mapsto \varphi_h : K \to K$
we can define

$$H \ltimes_\varphi K := \{(h,k) : h \in H, k \in K\}$$

with group structure

$$(h_1, k_1) * (h_2, k_2) := (h_1 h_2, \varphi_{h_2}(k_1) k_2)$$

Note: If $\varphi : H \to \mathrm{Aut}(K)$ is trivial i.e. $\varphi_h(k) = k$ $\forall h, k$. we say

$$H \ltimes_\varphi K = H \times K$$

"direct product"

Q: But groups = geometry. What does $\ltimes$ mean geometrically?

A: "Torsors"

Suppose group G acts on set S preserving some structure.

Easy but important Lemma:
For all $g \in G$, $x \in S$ we have

$$G_{g(x)} = g G_x g^{-1} \leq G$$

Proof: Given ghg^{-1} with $h(x) = x$, have

$$\begin{aligned} ghg^{-1}(g(x)) &= gh((g^{-1}g)(x)) \\ &= gh(x) \\ &= g(h(x)) \\ &= g(x) \implies ghg^{-1} \in G_{g(x)}. \end{aligned}$$

We have shown $g G_x g^{-1} \leq G_{g(x)}$.
Now let $x' = g(x)$, so $x = g^{-1}(x)$.

Same argument shows

$$(g^{-1}) G_{x'} (g^{-1})^{-1} \leq G_{g^{-1}(x')}$$
$$g^{-1} G_{g(x)} g \leq G_x$$
$$G_{g(x)} \leq g G_x g^{-1}$$

"Points in the same G-orbit have conjugate G-stabilizers"

If G acts "transitively" (i.e. $\forall x_1, x_2 \in S$ $\exists g \in G$ with $g(x_1) = x_2$) then for any $x \in S$ we get

$$S \longleftrightarrow G/G_x$$

We say G acts "freely" if $G_x = 1$ $\forall x \in S$ (i.e. $g(x) = h(x) \Rightarrow (g^{-1}h)(x) = x$ $\Rightarrow g^{-1}h \in G_x \Rightarrow g^{-1}h = 1 \Rightarrow g = h$).

Free + Transitive = Regular
(or "simply transitive")

If $G \curvearrowright S$ regularly then $\forall x \in S$ we have

$$S \longleftrightarrow G/G_x \longleftrightarrow G$$
$$g(x) \longleftrightarrow gG_x \longleftrightarrow g.$$

Then we can think of S as a group ($\approx G$) with identity element x.
But this is NOT CANONICAL!
It depends on the choice $x \in S$.

However, given any $x_1, x_2 \in S$ $\exists$ unique $g \in G$ with $g(x_1) = x_2$

Think: $g = $ "x_2/x_1"

and this is canonical.
We get a map $\theta : S \times S \longrightarrow G$
where $\theta(x_1, x_2) = $ "x_2/x_1"
= the unique g with $g(x_1) = x_2$

Observe $\theta(x,y) = 1 \implies 1(x) = y \implies x = y$.

And $\theta(y,z)\,\theta(x,y) = \theta(x,z)$ since
$g(x) = y$ and $h(y) = z$

$\implies (hg)(x) = h(g(x)) = h(y) = z$ //

Any choice of basepoint $x_0 \in S$ identifies S with G

$$\begin{array}{ccc} S & \overset{x_0}{\longleftrightarrow} & G \\ x & \longleftrightarrow & \theta(x_0, x) \end{array}$$

Q: Is S a group?

A: No. It's a "G-torsor"
(simple example of a principle G-bundle from gauge theory)

Axiomatically: Given group G, a G-torsor is a set S and a map $\theta : S \times S \to G$ such that $\forall x, y, z \in S$,

T1 $\theta(x, \cdot) : S \to G$ is surjective
T2 $\theta(x,y) = 1 \Rightarrow x = y$.
T3 $\theta(y,z)\,\theta(x,y) = \theta(x,z)$.

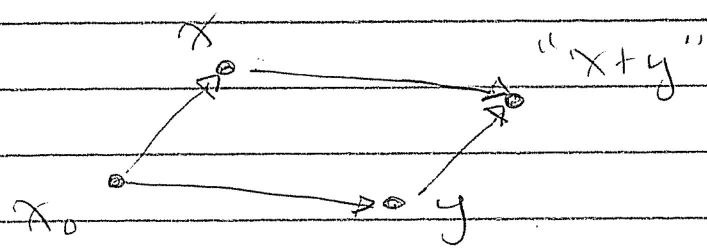
T1 + T2 + T3 also imply
T4 $\theta(x, \cdot) : S \to G$ is a bijection.

Note: Θ is equivalent to a regular group action $G \curvearrowright S$.

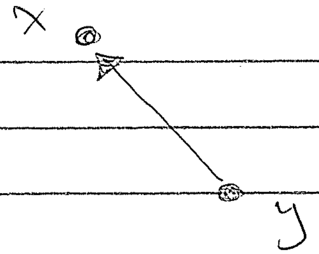
Examples:

- Energy is an $\mathbb{R}$-torsor.
  We can measure energy difference as a real number, but it makes no sense to discuss THE energy of a system.
- Voltage
- Indefinite integral
- 3D space is an $\mathbb{R}^3$-torsor

[you can only "add points" once you choose an arbitrary basepoint x_0



However, you can always "subract"



"$x-y$" exists, but it's not a point]

- G itself is a G-torsor with left regular action $g(h) := gh$.
I'll call this torsor

$\mathbb{A}G$ = "affine G"

↑
my notation

So what? Here's the point:

$$\mathrm{Aut}(\mathbb{A}G) \geq \mathrm{Aut}(G)$$

In fact we have

$$\mathrm{Aut}(\mathbb{A}G) \cong \mathrm{Aut}(G) \ltimes G$$

automorphisms ~~that fix the~~ basepoint.

change the basepoint

<u>Proof</u>: Given $g \in G$ define the translation map $\tau_g : AG \to AG$.

(1) Fix basepoint $x_0 \in G$

(2) Define

$$\begin{tikzcd} AG \arrow[r, leftrightarrow, "x_0"] \arrow[d, "\tau_g"'] & G \arrow[d] \arrow[r, phantom, "\ni"] & x \arrow[d, mapsto] \\ AG \arrow[r, leftrightarrow, "x_0"] & G \arrow[r, phantom, "\ni"] & gx \end{tikzcd}$$

(3) Check that $\tau_g : AG \to AG$ is independent of the choice $x_0 \in G$. (omitted).

Note that $\tau : G \longrightarrow \mathrm{Isom}(AG)$
$$g \longmapsto \tau_g$$
is an injective group hom:

(1) $\tau_g \circ \tau_h(x) = g(hx)$
$$= (gh)x = \tau_{gh}(x).$$

(2) If $gx = hx \quad \forall x \in G$.
Then $x = 1 \implies g = h$ ✓

Abuse notation and say

$$G \leq \mathrm{Isom}(\mathbb{A}G)$$

"translations"

Note $\tau_g \notin \mathrm{Aut}(G)$ if $g \neq 1$ since
$g(1) = g \neq 1$.

Given any $x_0 \in G$, get a subgroup

$$\mathrm{Aut}_{x_0}(\mathbb{A}G) \leq \mathrm{Aut}(\mathbb{A}G)$$

"$\varphi(x_0) = x_0$"

Note $\mathrm{Aut}_{x_0}(\mathbb{A}G) \cong \mathrm{Aut}(G)$ via $\theta(x_0, \circ)$

Might as well say $x_0 = 1$.

Have: $G \leq \mathrm{Aut}(\mathbb{A}G)$

$\mathrm{Aut}(G) \leq \mathrm{Aut}(\mathbb{A}G)$

with $G \cap \mathrm{Aut}(G) = \{id\}$

Now let $\varphi \in \mathrm{Aut}(\mathbb{A}G)$ and define
$g := \varphi(1)$. Then we have

$$\tau_{g^{-1}} \circ \varphi(1) = \tau_{g^{-1}}(g) = g^{-1}g = 1.$$

Hence $\mu := \tau_{g^{-1}} \circ \varphi \in \mathrm{Aut}(G)$.

$$\Longrightarrow \quad \varphi = \tau_g \circ \mu$$

$$\Longrightarrow \quad \mathrm{Aut}(AG) = G\,\mathrm{Aut}(G).$$

Finally note

$$\begin{aligned}
\mu \circ \tau_g(x) &= \mu(gx) \\
&= \mu(g)\mu(x) \\
&= \tau_{\mu(g)}(\mu(x)) \\
&= \tau_{\mu(g)} \circ \mu\,(x).
\end{aligned}$$

$$\Longrightarrow \quad \mu \circ \tau_g = \tau_{\mu(g)} \circ \mu.$$

OR $\boxed{\mu \circ \tau_g \circ \mu^{-1} = \tau_{\mu(g)}}$

We conclude that $G \trianglelefteq \mathrm{Aut}(AG)$, hence

$$\mathrm{Aut}(AG) = G \rtimes \mathrm{Aut}(G)$$

natural action

$\blacksquare$

This is the geometric meaning of $\rtimes$.