Thurs Oct 11

Recall: We say bilinear $B: V \times V \to \mathbb{F}$ is <u>non-degenerate</u> if the map

$$\begin{array}{ccc} V & \xrightarrow{\varphi_B} & V^* \\ x & \longmapsto & B(x, \cdot) \end{array}$$

is an isomorphism $V \cong V^*$. If V has basis $\beta = \{v_1, \dots, v_n\}$ then V^* has (dual) basis $\beta^* = \{v_1^*, \dots, v_n^*\}$ where

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Then in coordinates we can write

$$B(x, y) = [x]_\beta^t [B]_\beta [y]_\beta$$

and $[\varphi_B]_{\beta\beta^*} = [B]_\beta^t$, where

$$[B]_\beta = \begin{pmatrix} B(v_1, v_1) & B(v_1, v_2) & & \\ B(v_2, v_1) & B(v_2, v_2) & & \\ & & & \\ & & & B(v_n, v_n) \end{pmatrix}$$

the "Gram matrix"

If $\alpha \subseteq V$ is another basis then

$$[B]_\alpha = \underbrace{C_{\alpha\beta}^{t} [B]_\beta C_{\alpha\beta}}_{\text{NOT conjugation}}$$

Thus $\det [B]_\alpha \neq \det [B]_\beta$, so "$\det B$" is meaningless. However, since $\det C_{\alpha\beta} \neq 0$ we do have

$$\det [B]_\alpha = 0 \iff \det [B]_\beta = 0$$

Definition: $\forall x, y \in V$ we say

$$x \perp y \iff B(x,y) = 0$$

"orthogonal"

and for any subspace $W \subseteq V$ define

$$W^\perp := \{ x \in V : x \perp y \;\; \forall y \in W \}$$

"orthogonal complement"

Special Case: $V^\perp = \text{rad}(V) = \text{rad}(B)$

"the radical"

Theorem: Let $\dim(V, \mathbb{F}) < \infty$ and let $B : V \times V \to \mathbb{F}$ be bilinear. TFAE:

1. B is non-degenerate (i.e. $V \overset{\varphi_B}{\cong} V^*$)
2. $\det [B]_\beta = 0$ for any basis $\beta \subseteq V$
3. $V^\perp = \{0\}$ (i.e. $B(x,y) = 0 \ \forall y \Rightarrow x = 0$)

<u>Proof</u>: $(1) \Leftrightarrow (2)$

The map $\varphi_B : V \to V^*$ is an isomorphism $\Leftrightarrow \det \varphi_B \neq 0 \Leftrightarrow \det [B]_\beta \neq 0$, because $[\varphi_B]_{\beta\beta^*} = [B]_\beta^t$ ∎

$(1) \Leftrightarrow (3)$

Note:
$$\begin{aligned} V^\perp &= \ker \varphi_B. \\ &= \{x \in V : B(x,y) = 0 \ \forall y \in V\} \\ &= \{x \in V : B(x, \cdot) \in V^* \text{ is the zero map}\} \end{aligned}$$

Recall: "Rank-Nullity Theorem"
for linear $\varphi : U \to W$ we have

$$\dim \ker \varphi + \dim \operatorname{im} \varphi = \dim U$$
$$(\dim \ker \varphi_B + \dim \operatorname{im} \varphi_B = \dim V).$$

Thus $V^\perp = \ker \varphi_B = \{0\} \iff \varphi_B$ injective $\iff$

$\iff \dim \ker \varphi_B = 0$

$\iff \dim \operatorname{im} \varphi_B = \dim V$ (Rank-Nullity)

$\iff \dim \operatorname{im} \varphi_B = \dim V^*$ ($\dim V = \dim V^*$)

$\iff \operatorname{im} \varphi_B = V^*$

$\iff \varphi_B$ is surjective.

i.e. φ_B injective $\iff \varphi_B$ bijective

(3) (1)

$\boxtimes$

Recall: Forms $A, A' \in Mat_n(\mathbb{F})$ are <u>equivalent</u> if $\exists$ invertible $P \in GL_n(\mathbb{F})$ such that

$$A = P^t A' P$$

Q: Which forms A are equivalent to the standard form I ("dot product")?

A: $A = P^t I P = P^t P$ for some $P \in GL_n(\mathbb{F})$.

This implies

(1) $A^t = (P^t P) = P^t (P^t)^t = P^t P = A$.

("symmetric")

(2) $\det A = \det(P^t P) = \det(P)^2 \neq 0$
("non-degenerate").

In fact,

Theorem: If $\mathbb{F}$ is algebraically closed (or even just quadratically closed, i.e. every element has a square root) and if char $\mathbb{F} \neq 2$,

then EVERY symmetric non-degenerate form A is equivalent to I

Proof: Postponed..

More generally, we have

☆ Theorem: If char $\mathbb{F} \neq 2$, then EVERY symmetric bilinear form $B : V \times V \to \mathbb{F}$ can be diagonalized, i.e. $\exists$ basis $\beta \subseteq V$ such that

$$D = [B]_\beta$$

is diagonal.

Ingredients:

Given subspaces $U, W \subseteq V$. we say

$$V = U \oplus_B W \quad \text{"direct sum"} \quad \text{if}$$

① $V = U + W = \{x + y : x \in U, y \in W\}$
② $x \perp y$ (i.e. $B(x,y) = 0$) $\forall x \in U, y \in W$.

☆ Restate Theorem: Given symm. form $B : V \times V \to \mathbb{F}$ (char $\mathbb{F} \neq 2$) $\exists$ dim 1 subspaces $W_1, W_2, \ldots, W_n \subseteq V$ such that

$$V = W_1 \oplus_B W_2 \oplus_B \cdots \oplus_B W_n \quad ///$$

Lemma (Radical Splitting): Given $B : V \times V \to \mathbb{F}$ $\exists$ subspace $W \subseteq V$ such that

$$V = \text{rad}(V) \oplus_B W$$

and $B : W \times W \to \mathbb{F}$ is non-degenerate.

Proof: Choose any $W \subseteq V$ with $V = \text{rad}(V) + W$ and $\text{rad}(V) \cap W = \{0\}$.
(i.e. $V = \text{rad}(V) \oplus W$ in the usual sense)

Then $x \perp y$ $\forall$ $x \in rad(V)$, $y \in W$ by def of rad.
$\Longrightarrow$ $V =' rad(V) \oplus_B W$.

Finally note $rad(W) = W \cap rad(V) = \{0\}$
$\Longrightarrow$ B is non-deg. on W. $\blacksquare$

☆ Restate Theorem: $\forall$ symm. non-deg $B : V \times V \to \mathbb{F}$
we have $V = W_1 \oplus_B W_2 \oplus_B \cdots \oplus_B W_n$.

<u>Proof</u>: If B is degenerate, split the radical

$$\begin{aligned} V &= rad(V) \oplus W \\ &= \underbrace{(V_1 \oplus_B \cdots \oplus_B V_k)}_{\text{can choose ANY basis for the radical.}} \oplus (V_{k+1} \oplus_B \cdots \oplus_B V_n) \end{aligned}$$

$\blacksquare$

Lemma: For $B : V \times V \to \mathbb{F}$ non-deg. and subspace $W \subseteq V$ we have

$$\dim V = \dim W + \dim W^{\perp}$$

Proof: By non-deg., $\varphi_B : V \xrightarrow{\sim} V^*$.
Given subspace $\iota : W \hookrightarrow V$, we get a surjection of duals $\iota^* : V^* \twoheadrightarrow W^*$.
Compose:

$$\iota^* \circ \varphi_B : V \xrightarrow{\sim} V^* \twoheadrightarrow W^*$$

Rank-Nullity says

$$\dim \ker(\iota^* \circ \varphi_B) + \dim \operatorname{im}(\iota^* \circ \varphi_B) = \dim V$$
$$\dim W^{\perp} \qquad + \qquad \dim W^* \qquad = \dim V.$$

But $\dim W^* = \dim W$

$$\Longrightarrow \dim V = \dim W + \dim W^{\perp}.$$

WARNING: This doesn't imply
$$V = W \oplus_B W^{\perp}$$

Example: Consider form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{R}^2$

$$(1, 1) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

Hence $B\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = 0$

(we say that $\begin{pmatrix}1\\1\end{pmatrix}$ is "isotropic")

By dimension count we know

$$\operatorname{span}\begin{pmatrix}1\\1\end{pmatrix}^{\perp} = \operatorname{span}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Longrightarrow \quad \mathbb{R}^2 \neq \operatorname{span}\begin{pmatrix}1\\1\end{pmatrix} \oplus_B \operatorname{span}\begin{pmatrix}1\\1\end{pmatrix}^{\perp}.$$

However,

Lemma: If $x \in V$ is not isotropic (i.e. $B(x,x) \neq 0$) then $V = \operatorname{span}(x) \oplus_B \operatorname{span}(x)^{\perp}$.

Proof: $B(x,x) \Longrightarrow x \notin \operatorname{span}(x)^{\perp}$.
Done by dimension count ∎

Finally we can prove the Theorem.

Tues Oct 16

Structure of Bilinear Forms.

Recall: Given bilinear $B : V \times V \to \mathbb{F}$ we say

$x \perp_B y \iff B(x,y) = 0$
"orthogonal"

Given subspaces $U, W \subseteq V$ we say

$V = U \oplus_B W$ "orthogonal direct sum"

if ① $V = U + W$
$= \{ x + y : x \in U, y \in V \}$
② $x \perp_B y \;\; \forall x \in U, y \in W$

[Remark: This conflicts slightly with the traditional direct sum

$$V = U \oplus W \iff \begin{cases} V = U + W \\ U \cap W = \{0\} \end{cases}$$
]

Given subspace $W \subseteq V$ let

$W^\perp := \{ x \in V : x \perp y \;\; \forall y \in W \}$.
"orthogonal complement"

Special Case:

$$V^{\perp} = \mathrm{rad}(V) = \mathrm{rad}(B).$$
"the radical of B".

Recall: B is non-degenerate $\iff V^{\perp} = \{0\}$.

DEF: We say $B : V \times V \to \mathbb{F}$ is symmetric if $B(x,y) = B(y,x) \ \forall x,y \in V$.

Equivalently, $B(x,y) = x^t A y$ where $A \in \mathrm{Mat}_n(\mathbb{F})$ is symmetric $(A^t = A)$.

☆ Structure Theorem ☆:
If $\mathrm{char}\,\mathbb{F} \neq 2$, then Every symmetric bilinear $B : V \times V \to \mathbb{F}$ can be "diagonalized", i.e. $\exists$ basis $\beta \subseteq V$ such that

$$D = [B]_{\beta} \text{ is diagonal.}$$

Equivalently, $\exists$ 1-dim subspaces $W_1, W_2, \ldots, W_n \subseteq V$ such that

$$V = W_1 \oplus_B W_2 \oplus_B \cdots \oplus_B W_n$$

//.

We need 4 Lemmas.

Lemma 1 (Radical Splitting): Given bilinear $B: V \times V \to \mathbb{F}$ $\exists$ subspace $W \subseteq V$ such that

$$V = \mathrm{rad}(V) \oplus_B W$$

and $B: W \times W \to \mathbb{F}$ is <u>non-degenerate</u>.

<u>Proof</u>: Choose any $W \subseteq V$ with $V = \mathrm{rad}(V) + W$ and $\mathrm{rad}(V) \cap W = \{0\}$.
Then $V = \mathrm{rad}(V) \oplus_B W$.

Clearly $\mathrm{rad}(W) \supseteq W \cap \mathrm{rad}(V)$.
But we also have $\mathrm{rad}(W) \subseteq W \cap \mathrm{rad}(V)$
since if $B(x,y) = 0$ $\forall y \in W$ then
$V = \mathrm{rad}(V) + W \implies \forall y \in V$ we can
write $y = y_0 + y_1$, $y_0 \in \mathrm{rad}(V)$, $y_1 \in W$.
Then

$$B(x,y) = B(x,y_0) + B(x,y_1)$$
$$= 0 + 0 = 0.$$

Hence $\mathrm{rad}(W) = W \cap \mathrm{rad}(V) = \{0\}$

$\blacksquare$

Lemma 2: If $B: V \times V \to \mathbb{F}$ is <u>non-deg</u>. then $\forall$ subspaces $W \leq V$ we have

$$\dim V = \dim W + \dim W^{\perp}$$

<u>Proof</u>: Since B is non-degenerate we have an isomorphism $\varphi_B : V \xrightarrow{\sim} V^*$. The inclusion $\iota : W \hookrightarrow V$ induces a surjection of duals $\iota^* : V^* \twoheadrightarrow W^*$. Compose:

$$\iota^* \circ \varphi_B : V \xrightarrow{\sim} V^* \twoheadrightarrow W^*$$

Rank-Nullity says

$$\dim \ker(\iota^* \circ \varphi_B) + \dim \operatorname{im}(\iota^* \circ \varphi_B) = \dim V$$
$$\dim W^{\perp} + \dim W^* = \dim V$$
$$\dim W^{\perp} + \dim W = \dim V$$

$\blacksquare$

WARNING: This <u>doesn't</u> imply $V = W \oplus_B W^{\perp}$

eg. Consider form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{R}^2$.

$$(1, 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$$

Hence $B\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = 0$
(say $\begin{pmatrix}1\\1\end{pmatrix}$ is "isotropic").

By dimension count we know

$$\operatorname{span}\begin{pmatrix}1\\1\end{pmatrix}^{\perp} = \operatorname{span}\begin{pmatrix}1\\1\end{pmatrix}$$

But $\mathbb{R}^2 \neq \operatorname{span}\begin{pmatrix}1\\1\end{pmatrix} \oplus_B \operatorname{span}\begin{pmatrix}1\\1\end{pmatrix}$!

However,
Lemma 3: If $x \in V$ is not isotropic (i.e. $B(x,x) \neq 0$) then

$$V = \operatorname{span}(x) \oplus_B \operatorname{span}(x)^{\perp}$$

Proof: $B(x,x) \neq 0 \implies x \notin \operatorname{span}(x)^{\perp}$.
Done by dimension count (Lemma 2) ∎

Lemma 4: If $B: V \times V \to \mathbb{F}$ is non-deg and symmetric then $\exists\, x \in V$ with $B(x,x) \neq 0$.

Proof: By non-degeneracy $\exists\, u, w \in V$ with $B(u,w) \neq 0$. If $B(u,u) \neq 0$ or $B(w,w) \neq 0$ we're done. Otherwise let $x := u + w$. Then

$$\begin{aligned} B(x,x) &= B(u+w, u+w) \\ &= \cancel{B(u,u)} + B(u,w) + B(w,u) + \cancel{B(w,w)} \\ &= 2B(u,w) \neq 0 . \end{aligned}$$

$\blacksquare$

☆ Finally we can prove the Structure Theorem: Every symmetric form is diagonalizable.

Proof: By radical splitting, it is enough to show this for non-degenerate B since

$$\begin{aligned} V &= \mathrm{rad}(V) \oplus_B W \\ &= \underbrace{(W_1 \oplus_B \cdots \oplus_B W_k)}_{\text{choose ANY basis for } \mathrm{rad}(V).} \oplus_B (W_{k+1} \oplus_B \cdots \oplus_B W_n) \end{aligned}$$

So suppose $B: V \times V \to \mathbb{F}$ is symmetric and non-degenerate. By Lemma 4 $\exists\, x \in V$ with $B(x,x) \neq 0$

By Lemma 3 we have

$$V = \mathrm{span}(x) \oplus_B \mathrm{span}(x)^{\perp}$$

with B symmetric and non-degenerate on $\mathrm{span}(x)^{\perp}$.

We're done by induction $\blacksquare$

Summary: Given <u>any</u> sym<u>metric</u> matrix $A \in \mathrm{Mat}_n(\mathbb{F})$ $\exists$ $P \in GL_n(\mathbb{F})$ such that

$$P^t A P = \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right)$$

where D is <u>diagonal</u>.