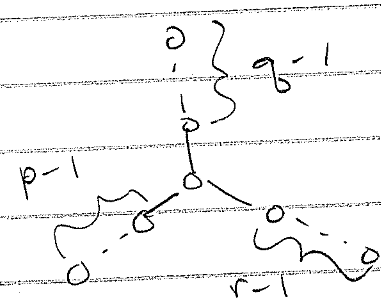
Tues Oct 2

Today: Numerology and a Game

Recall the "triangle graph"

$T_{p,q,r}$ =



with $p+q+r-2$ vertices.

Recall the
☆ Very Big Theorem ☆:

Let G be any simple undirected graph. Then

(1) $\rho(G) < 2 \iff G = T_{p,q,r}$ with

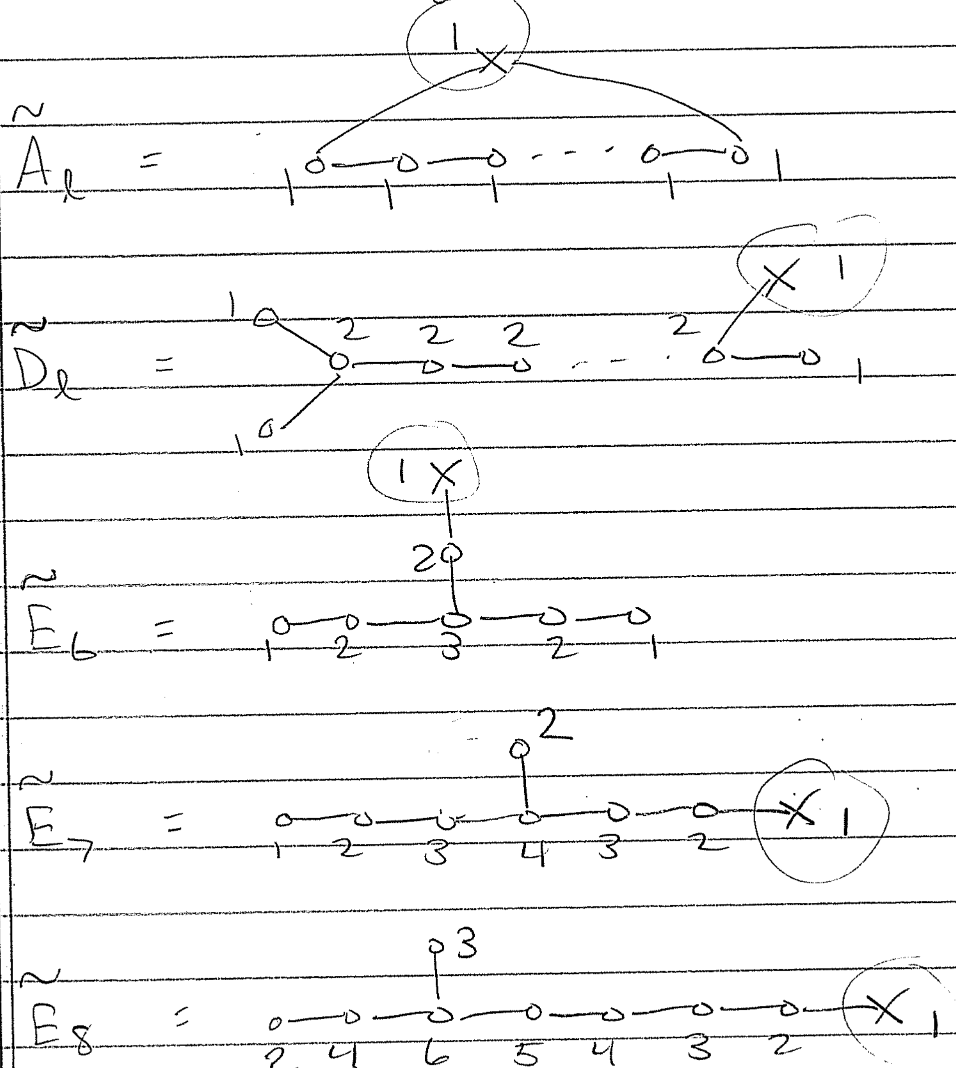
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

These graphs have special names:

$\{p,q,r\}$	$1,1,\ell$	$2,2,\ell-2$	$2,3,3$	$2,3,4$	$2,3,5$
G	A_ℓ	D_ℓ	E_6	E_7	E_8

"Coxeter graphs" of finite type.

Each of these graphs has an "affine extension"



(2) These graphs have $\rho(G) = 2$ with displayed "special" PF vector

and these are the <u>only</u> graphs with $\rho(G) = 2$

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So what?

For the rest of the year, I will try to convince you this is interesting.

Some Numerology:

let $G \in \left\{ T_{p,q,r} : \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 \right\}$

and define

$h_G :=$ <u>sum</u> the components of the special PF vector of $\tilde{G}$.

$=$ the "Coxeter number" of G.

i.e.

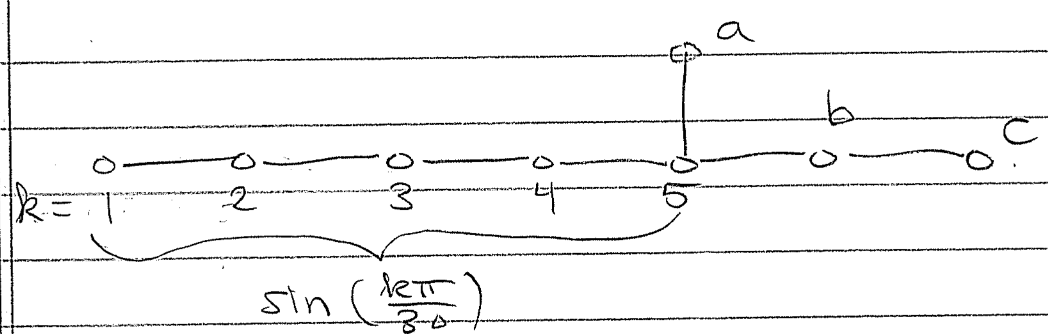
G	A_ℓ	D_ℓ	E_6	E_7	E_8
h_G	$\ell+1$	$2(\ell-1)$	12	18	30

Theorem:

$$\rho(G) = 2\cos\left(\frac{\pi}{h_G}\right).$$

Problem: Express $h_{T_{p,q,r}}$ as a function of p, q, r.
(I don't know the answer.)

eg. Here is the PF vector of E_8 with evalue $2\cos(\pi/30)$.



$$a = \sin\left(\frac{5\pi}{30}\right) \Big/ 2\cos\left(\frac{\pi}{30}\right)$$

$$b = \sin\left(\frac{7\pi}{30}\right) \Big/ 2\cos\left(\frac{\pi}{30}\right)$$

$$c = \sin\left(\frac{7\pi}{30}\right) \Big/ \left(2\cos\left(\frac{\pi}{30}\right)\right)^2$$

source: Goodman - de la Harpe - Jones.

Q: What does this mean?
(Recall Lagrange and type A_ℓ).

A: See article "Did a 1-dimensional magnet detect a 248-Dimensional Lie Algebra?" by Borthwick - Garibaldi.

What about the other eigenvalues?

Let G_ℓ be a finite type Coxeter graph. The eigenvalues of G_ℓ are

$$2\cos\left(\frac{(d_i-1)\pi}{h_G}\right) \quad i=1,2,\dots,\ell$$

Where $d_1 \le d_2 \le \cdots \le d_\ell$ are very special integers called the "degrees" of G_ℓ.

- Since $\rho(G_\ell) = 2\cos(\pi/h_G)$ we know that the smallest degree is $d_1 = 2$

- Since G_ℓ is bipartite, we know that

$$2\cos\left(\frac{(d_\ell-1)\pi}{h_G}\right), \dots, 2\cos\left(\frac{(d_1-1)\pi}{h_G}\right)$$

is symmetric about zero. That is, for all i we have

$$2\cos\left(\frac{(d_i-1)\pi}{h_G}\right) = -2\cos\left(\frac{(d_{\ell-i+1}-1)\pi}{h_G}\right)$$

$$= 2\cos\left(\pi - \frac{(d_{\ell-i+1}-1)\pi}{h_G}\right)$$

$$= 2\cos\left(\frac{[(h_G-d_{\ell-i+1}+2)-1]\pi}{h_G}\right).$$

$$\Longrightarrow \quad d_i = h_G - d_{\ell-i+1} + 2.$$

In particular, the largest degree is

$$d_\ell = h_G - d_1 + 2$$
$$= h_G - 2 + 2 = h_G$$

Table of Degrees.

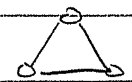
A_ℓ	$2, 3, 4, \ldots, \ell+1$.
D_ℓ	$2, 4, 6, \ldots, 2(\ell-1), \ell$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$.

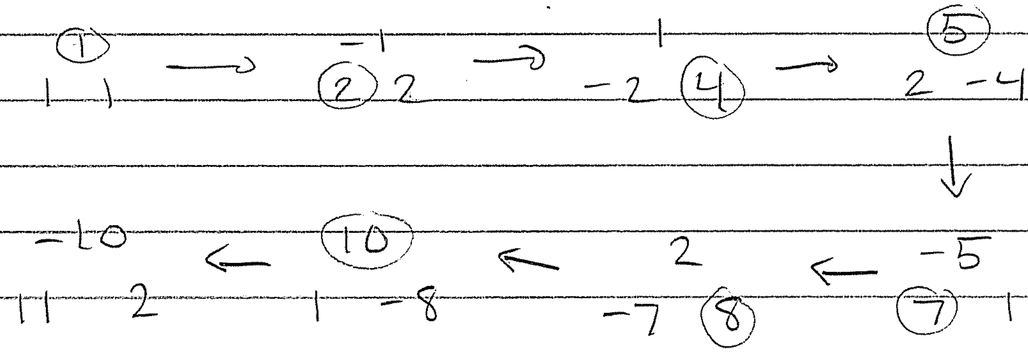
So what?

OK, here's a game.

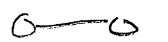
- Let G be any connected simple graph.
- A "state" is a labeling of the vertices v_i by numbers $w(v_i)$
- To "fire" the vertex v_i:
  - replace $w(v_i)$ by $-w(v_i)$
  - if $v_i \leftrightarrow v_j$ are adjacent, replace $w(v_j)$ by $w(v_j) + w(v_i)$.

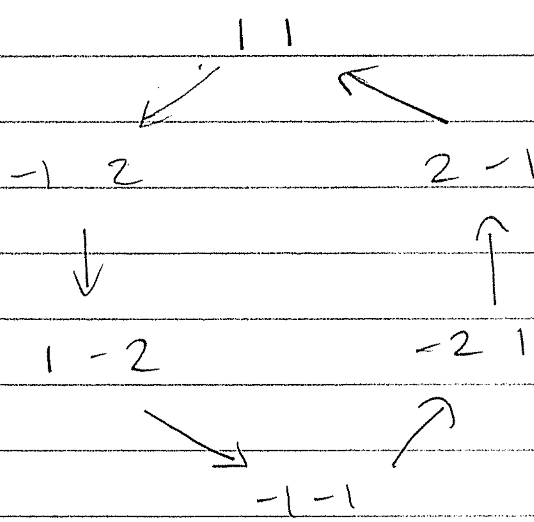
The Game: Start with a generic configuration, say $w(v_i) = 1 \ \forall i$. Fire the vertices in any order you choose.

eg $G =$ 



This could go on forever.

eg $G =$ 



The game loops back

eg $G = \circ - \circ - \circ$.

(see handout. It's finite with 24 states).

Theorem:

The game is finite if and only if G is a finite type Coxeter graph, in which case.

$$\#\text{states} = \underbrace{d_1 d_2 \cdots d_\ell}_{\text{product of the "degrees"}}.$$

eg. The E_8 game is finite with

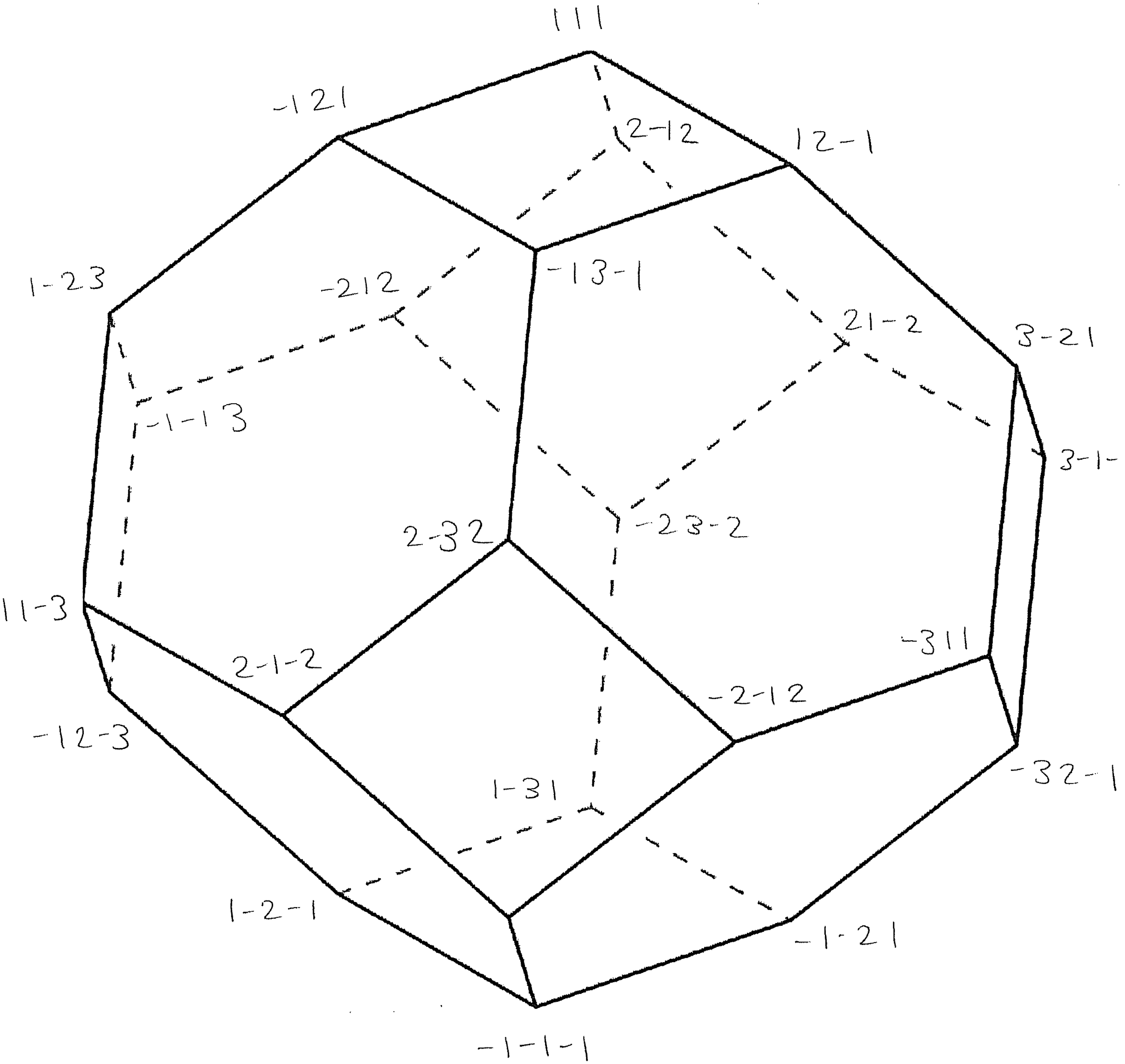
$$2 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \cdot 20 \cdot 24 \cdot 30$$

$$= 696{,}729{,}600 \text{ states.}$$

Proof: I.O.U.

Hint: The game is a group.
(a "Coxeter group")

eg. G = o—o—o


111
-121
2-12
12-1
1-23
-212
-13-1
21-2
3-21
-1-13
3-1-1
2-32
-23-2
11-3
-311
2-1-2
-2-12
-12-3
-32-1
1-31
1-2-1
-1-21
-1-1-1