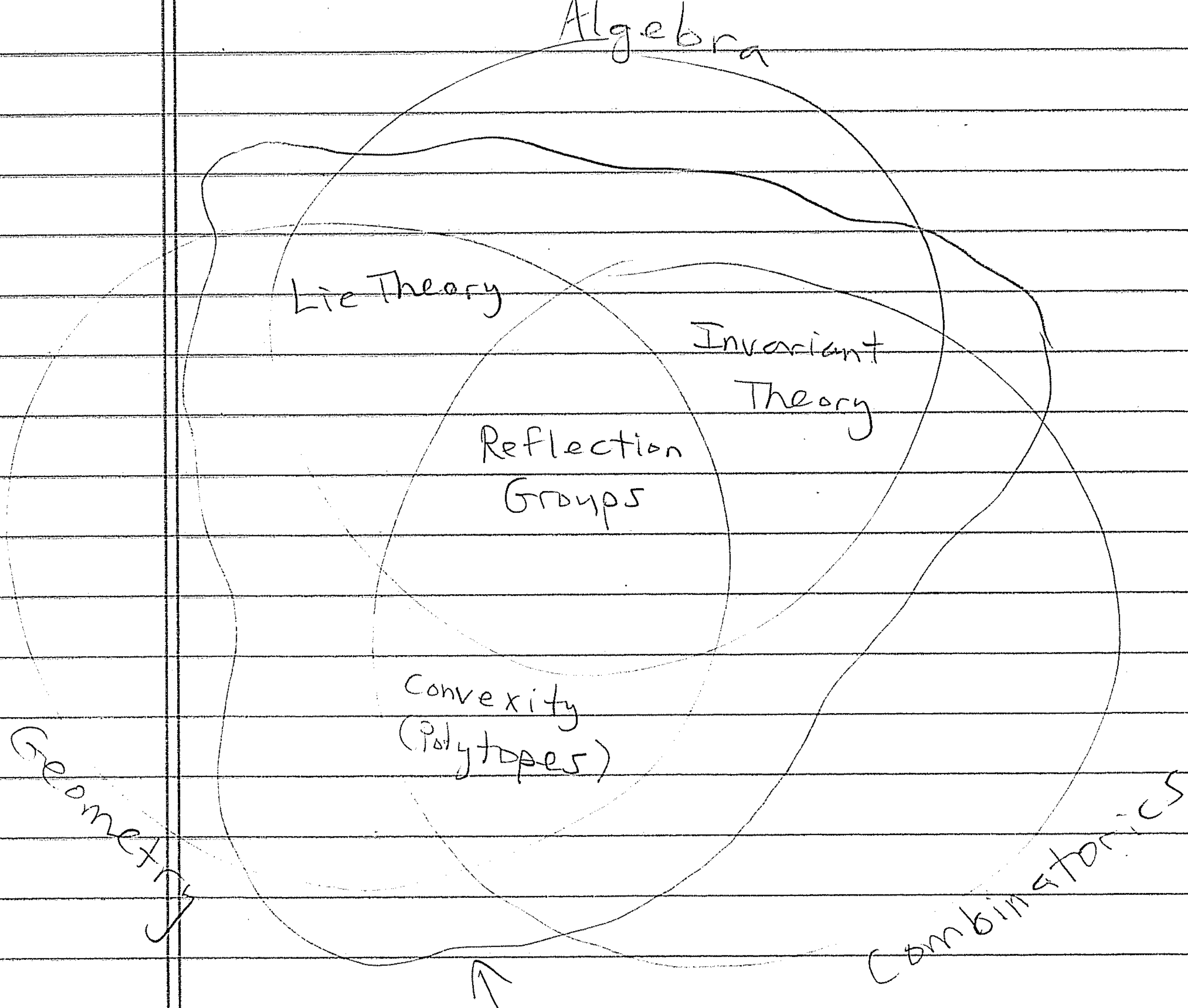
Thurs, Sept 23
2012

MTH 592/685 (Topics)
"Reflection Groups"

Sketch of the Course:



Linear Algebra
(I'm not kidding)

Timeline:

- Euclid, ~300B
  - classified the regular polyhedra (3D)

tetrahedron	cube	octahedron	icos.	dodec.
fire	earth	air	water	Aether

- Schläfli, ~1850
  - classified the regular "solids" in all dimensions (!)
  - forgotten and rediscovered ≥ 9 times
- Killing-Cartan, ~1890
  - classified "infinitesimal rigid motions" in any type of space (Lie algebras)
- Coxeter, ~1935
  - classified finite groups generated by reflections (Coxeter groups)
- Chevalley, Shephard-Todd, ~1955
  - classified groups whose invariant ring is a free polynomial algebra.

- post-1955, stay tuned!

Surprise: These classifications are all "the same". Described by the Coxeter diagrams.

		Regular Polytope	Lie algebra
A_n	o—o—o ··· o—o	simplex	$SL(n+1)$
B_n	o—o—o ··· o—o $\xrightarrow{4}$ o	cube	$SO(2n+1)$
C_n	o—o—o ··· o—o $\xleftarrow{4}$ o	octahedron	$Sp(n)$
D_n	o—o—o ··· o—o (extra o attached to the 2nd node)	X	$SO(2n)$
E_6	o—o—o—o—o (extra o attached to the 3rd node)	X	E_6
E_7	o—o—o—o—o—o (extra o attached to the 3rd node)	X	E_7
E_8	o—o—o—o—o—o—o (extra o attached to the 3rd node)	X	E_8
F_4	o—o $\overset{4}{—}$ o—o	24-cell	F_4
$G_2(m)$	o $\overset{m}{—}$ o	m-gon	$G_2 = G_2(6)$
H_3	o—o $\overset{5}{—}$ o	icos./dodec.	X
H_4	o—o—o $\overset{5}{—}$ o	120-cell/600-cell	X

Homework: Read Coleman's paper,
"The Greatest Mathematical Paper of All Time", Math. Int. (1989)

1. Euclid, "Elements"
2. Newton, "Principia..."
3. W. Killing, "Z. v. G. II", (1888)

- the paradigm for mathematical classification.

Problem: Given an abstract mathematical structure, find/construct/classify all specific examples.

- Killing gave hope that this is possible.
- classification is a shortcut to further progress / general theorems

BEGIN.

Topic: Linear Algebra
Example of "classification"

Q: What is "space"?

A: A "vector space" is a pair $(V, \mathbb{F})$ where

- $(V, +, \vec{0})$ is an abelian group (of "vectors")
- $(\mathbb{F}, 0, 1)$ is a field (of "scalars")

such that $\mathbb{F}$ "acts linearly" on V

by $$\mathbb{F} \times V \longrightarrow V$$
$$(a, \vec{v}) \longmapsto a\vec{v}$$

i.e. $\forall\, \vec{v} \in V$, $a, b \in \mathbb{F}$ we have

- $1\vec{v} = \vec{v}$
- $(ab)\vec{v} = a(b\vec{v})$
- $(a+b)\vec{v} = a\vec{v} + b\vec{v}$ //

<u>Example</u>: Given field $\mathbb{F}$, integer $n \geq 1$, let

$$V = \mathbb{F}^n := \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \ldots, a_n \in \mathbb{F} \right\}$$

with

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} := \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} \quad \& \quad r\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} ra_1 \\ \vdots \\ ra_n \end{pmatrix}$$

//

Def: Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in (V, \mathbb{F})$ are called independent if

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}$$

$$\implies a_1 = a_2 = \cdots = a_n = 0$$

(no linear relationships among $\vec{v}_i$)

Nontrivial (!) Fact: If $(V, \mathbb{F})$ is finitely generated (i.e. $\exists\, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in V$ such that

$$V = \{a_1\vec{v}_1 + \cdots + a_k\vec{v}_k : a_1, a_2, \ldots, a_k \in \mathbb{F}\})$$

then every maximal independent subset $B \subseteq V$ has the same finite size, called the dimension of $(V, \mathbb{F})$.

$$\dim(V) := |B|.$$

B is called a basis for $(V, \mathbb{F})$.

Proof: c.f. the theory of MATROIDS

$\boxtimes$

Classification Theorem: If $(V, \mathbb{F})$ has dimension $n < \infty$, then

$$(V, \mathbb{F}) \approx \mathbb{F}^n$$

[- i.e. "f.d. vector space" = "(field, number)"
- If the field is assumed, then "f.d. vector space" = "a number".]

<u>Proof</u>: let $\vec{e}_1, \ldots, \vec{e}_n \in V$ be <u>any</u> basis. Define a map $\varphi : V \to \mathbb{F}^n$ by

$$\varphi(\vec{e}_i) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{- ith position}$$

and extend linearly, i.e.

$$\varphi(a_1\vec{e}_1 + \cdots + a_n\vec{e}_n) := a_1\varphi(\vec{e}_1) + \cdots + a_n\varphi(\vec{e}_n)$$

$$= a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + a_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

This is an isomorphism. $\boxtimes$