6/5/14

Review of 661/662

We have seen:

① Abstract Groups
② Groups Acting on Things (Klein)
③ Abstract Rings

Today: Rings of Functions

Let X be a "space". In Klein's philosophy we study X via group actions $G \curvearrowright X$. In Grothendieck's philosophy we study X via functions $f: X \to K$ into a field K. These functions form a <u>ring</u> under pointwise operations

- $(f+g)(x) := f(x) + g(x)$
- $(fg)(x) := f(x)g(x)$.

Conversely, if R is an abstract ring we try to think of it as a ring of functions on <u>some</u> "space" X.

The prototypical example is the ring of "polynomial" functions on K^n when K is algebraically closed. Hilbert's NSS gives us a correspondence:

$K[x_1, \dots, x_n]$	K^n
radical ideals	varieties
prime ideals	irreducible varieties
maximal ideals	points

Instead of pursuing scheme theory we go back to the beginning.

Definition: Let $I \subseteq R$ be an ideal. We say

- I is radical if $f^n \in I \Rightarrow f \in I$.
- I is prime if $fg \in I \Rightarrow f \in I$ or $g \in I$.
- I is maximal if $I \subsetneq J \subseteq R \Rightarrow J = R$.

Exercise:

- I radical $\Leftrightarrow$ R/I reduced (no nilpotents)
- I prime $\Leftrightarrow$ R/I domain (no zero divisors)
- I maximal $\Leftrightarrow$ R/I field.

Corollary :

maximal $\Rightarrow$ prime $\Rightarrow$ radical. ///

The Krull dimension d of R is the length of the longest chain of prime ideals

$$P_0 < P_1 < P_2 < \cdots < P_d < R$$

(generalization of vector space dimension)

Examples :

- fields have dim 0
- PIDs have dim 1.

Let R be a PID and consider the ring of polynomials $R[x]$.

If $R[x]$ is a PID then R is a field.

Proof: Clearly the map $a \mapsto a + (x)$ is a surjective ring hom $R \twoheadrightarrow R[x]/(x)$. Since degree is additive (R is a domain)

$\wr$

it is also injective, so we have a ring isom.

$$R \approx R[x]/(x).$$

I claim that $(x) < R[x]$ is maximal.
If there exists $(x) < I < R$ then since R is a PID we have $I = (f(x))$ and hence $x = f(x)g(x)$ where f, g are not units. Since R is a domain this implies

$$1 = \deg(x) \geq \deg(f) + \deg(g) \geq 1 + 1 = 2.$$

Contradiction. Hence (x) is maximal and $R \approx R[x]/(x)$ is a field.

Corollary: $\mathbb{Z}[x]$ and $K[x,y]$ are <u>not</u> PIDs.

However we do have

Gauss' Lemma:

$\mathbb{Z}[y]$ and $K[x,y]$ are UFDs.

(More generally:

$$R \text{ UFD} \implies R[x] \text{ UFD} \quad)$$

Proof: Let R be PID. We say $f(x) \in R[x]$ is *primitive* if gcd(coefficients) $= 1$.
Given prime $p \in R$ we consider the reduction

$$R[x] \twoheadrightarrow R/(p)[x].$$
$$f(x) \longmapsto \bar{f}(x).$$

Given $f, g \in R[x]$ primitive, assume fg not primitive, i.e., $0 = \overline{fg}$ for some p. But then $\bar{f}\bar{g} = \overline{fg} = 0$ implies $\bar{f} = 0$ or $\bar{g} = 0$. Contradiction. Hence fg is prim.

$=$

Now let $K = \mathrm{Frac}(R)$ and consider any polynomial $f(x) \in R[x]$. Since $K[x]$ is Euclidean (hence PID, hence UFD) we can write

$$f(x) = \alpha_1 f_1(x) \cdots f_l(x)$$

where $\alpha \in K^\times$ and $f_i(x)$ are irreducible $/K$. We write $f_i(x) = \alpha_i q_i(x)$ where $\alpha_i \in K$ and $q_i(x) \in R[x]$ is primitive, so

$$f(x) = \underbrace{\alpha\alpha_1 \cdots \alpha_\ell}_{\in K^\times} \underbrace{g_1(x) \cdots g_\ell(x)}_{\text{primitive} \in R[x]}$$

Now use the fact that R is PID to show that $\beta := \alpha\alpha_1 \cdots \alpha_\ell \in R$ (Exercise). Factor β in R to get

$$f(x) = u\, p_1 \cdots p_k\, g_1(x) \cdots g_\ell(x).$$

unit irred constants irred polynomials.

Thus every $f \in R[x]$ can be factored.

Next we must show irred $\Rightarrow$ prime in $R[x]$. We do this in two steps.

Let $p \in R[x]$ be an irred constant. If $p \mid g(x)h(x)$ then

$$0 = \overline{gh} = \overline{g}\,\overline{h} \Rightarrow \overline{g} = 0 \text{ OR } \overline{h} = 0$$
$$(p \mid g(x)) \quad (p \mid h(x))$$

hence p is prime in $R[x]$.

Let $g(x) \in R[x]$ be irred. (hence primitive) polynomial. Then $g(x)$ is also irred in $K[x]$. (Proof: If $q(x) = g(x)h(x)$ in $K[x]$ we write $q(x) = \alpha g_1(x) \beta h_1(x)$ $= \alpha\beta g_1(x) h_1(x)$ where $g_1(x) h_1(x) \in R[x]$ is primitive. Since R is PID we have again $\alpha\beta \in R$. Hence q is reducible over R.) Since $q(x)$ is irred in the PID $K[x]$, Euclid's Lemma implies $q(x)$ is prime in $K[x]$. But then $q(x)$ is also prime in $R[x]$ since if $q \mid gh$ over R then $q \mid gh$ over K $\Rightarrow$ $q \mid g$ OR $q \mid h$ over K. Using the same trick one more time we get $q \mid g$ OR $q \mid h$ over R.

Hence $q(x)$ is prime in $R[x]$.

$=$

We conclude that $R[x]$ is a UFD.

///

Building on this we can prove the following theorem (with a lot of work).

Theorem: Let R be PID. Then the prime ideals of $R[x]$ are exactly

- (0)
- $(q(x))$ for irreducible $q(x) \in R[x]$
- $(p, f(x))$ where $p \in R$ is prime and $\bar{f}(x) \in R/(p)[x]$ is irreducible.

The third kind are the maximal ideals.

Corollary: $R[x]$ has Krull dim 2

(In general, $\dim(R[x]) \geq \dim(R) + 1$).

Remark: Galois connections

$$\xrightarrow{I} \quad \xleftarrow[V]{}$$

will NOT be on the exam.