

6/5/14

Review of 661/662

We have seen:

- ① Abstract Groups
- ② Groups Acting on Things (Klein)
- ③ Abstract Rings

Today: Rings of Functions

Let X be a "space". In Klein's philosophy we study X via group actions $G \curvearrowright X$.

In Grothendieck's philosophy we study X via functions $f: X \rightarrow K$ into a field K . These functions form a ring under pointwise operations

- $(f+g)(x) := f(x) + g(x)$
- $(fg)(x) := f(x)g(x)$.

Conversely, if R is an abstract ring we try to think of it as a ring of functions on some "space" X .

The prototypical example is the ring of "polynomial" functions on K^n when K is algebraically closed. Hilbert's NSS gives us a correspondence:

$K[x_1, \dots, x_n]$	K^n
radical ideals	varieties
prime ideals	irreducible varieties
maximal ideals	points

Instead of pursuing scheme theory we go back to the beginning.

Definition: Let $I \subseteq R$ be an ideal. We say

- I is radical if $f^n \in I \Rightarrow f \in I$.
- I is prime if $fg \in I \Rightarrow f \in I \text{ OR } g \in I$.
- I is maximal if $I \subsetneq J \subseteq R \Rightarrow J = R$.

Exercise:

- I radical $\Leftrightarrow R/I$ reduced (no nilpotents)
- I prime $\Leftrightarrow R/I$ domain (no zero divisors)
- I maximal $\Leftrightarrow R/I$ field.

it is also injective, \Rightarrow we have a ring isom.

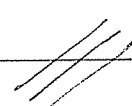
$$R \cong R[x]/(x).$$

I claim that $(x) \subset R[x]$ is maximal.

If there exists $(x) \subset I \subset R[x]$ then since R is a PID we have $I = (f(x))$ and hence $x = f(x)g(x)$ where f, g are not units.

Since R is a domain this implies

$$1 = \deg(x) \geq \deg(f) + \deg(g) \geq 1 + 1 = 2.$$

Contradiction. Hence (x) is maximal and $R \cong R[x]/(x)$ is a field. 

Corollary: $\mathbb{Z}[x]$ and $K[x, y]$ are not PIDs.

However we do have

Gauss' Lemma:

$\mathbb{Z}[y]$ and $K[x, y]$ are UFDs.

(More generally:

$$R \text{ UFD} \implies R[x] \text{ UFD})$$

Proof: Let R be PID. We say $f(x) \in R[x]$ is primitive if $\gcd(\text{coefficients}) = 1$.

Given prime $p \in R$ we consider the reduction

$$\begin{aligned} R[x] &\longrightarrow R/(p)[x] \\ f(x) &\longmapsto \bar{f}(x). \end{aligned}$$

Given $f, g \in R[x]$ primitive, assume fg not primitive, i.e., $0 = \bar{f}\bar{g}$ for some p . But then $\bar{f}\bar{g} = \bar{f}\bar{g} = 0$ implies $\bar{f} = 0$ or $\bar{g} = 0$. Contradiction. Hence fg is prim.

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Now let $K = \text{Frac}(R)$ and consider any polynomial $f(x) \in R[x]$. Since $K[x]$ is Euclidean (hence PID, hence UFD) we can write

$$f(x) = \alpha \cdot f_1(x) \cdots f_t(x)$$

where $\alpha \in K^\times$ and $f_i(x)$ are irreducible $/K$. We write $f_i(x) = \alpha_i g_i(x)$ where $\alpha_i \in K$ and $g_i(x) \in R[x]$ is primitive, so

$$f(x) = \underbrace{\alpha \alpha_1 \dots \alpha_l}_{\in K^*} \underbrace{g_1(x) \dots g_e(x)}_{\text{primitive} \in \mathbb{R}[x]}$$

Now use the fact that R is PID to show that $\beta := \alpha \alpha_1 \dots \alpha_l \in R$ (Exercise). Factor β in R to get

$$f(x) = \underbrace{u}_{\text{unit}} \underbrace{p_1 \dots p_k}_{\substack{\uparrow \\ \text{irred} \\ \text{constants}}} \underbrace{g_1(x) \dots g_e(x)}_{\substack{\uparrow \\ \text{irred} \\ \text{polynomials}}}$$

Thus every $f \in R[x]$ can be factored.

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Next we must show $\text{irred} \Rightarrow \text{prime}$ in $R[x]$. We do this in two steps.

Let $p \in R[x]$ be an irred constant.

If $p \mid g(x)h(x)$ then

$$0 = \overline{g}h = \overline{g} \overline{h} \Rightarrow \overline{g} = 0 \text{ OR } \overline{h} = 0$$

$(p \mid g(x)) \quad (p \mid h(x))$

hence p is prime in $R[x]$.

Let $g(x) \in R[x]$ be irred. (hence primitive) polynomial. Then $g(x)$ is also irred in $K[x]$. (Proof: If $g(x) = g_1(x)h_1(x)$ in $K[x]$ we write $g(x) = \alpha g_1(x) \beta h_1(x) = \alpha \beta g_1(x)h_1(x)$ where $g_1(x)h_1(x) \in R[x]$ is primitive. Since R is PID we have again $\alpha \beta \in R$. Hence g is reducible over R .) Since $g(x)$ is irred in the PID $K[x]$, Euclid's Lemma implies $g(x)$ is prime in $K[x]$. But then $g(x)$ is also prime in $R[x]$ since if $g \mid gh$ over R then $g \mid gh$ over $K \implies g \mid g$ or $g \mid h$ over K . Using the same trick one more time we get $g \mid g$ or $g \mid h$ over R .

Hence $g(x)$ is prime in $R[x]$.

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We conclude that $R[x]$ is a UFD.



Building on this we can prove the following theorem (with a lot of work).

Theorem: Let R be PID. Then the prime ideals of $R[x]$ are exactly

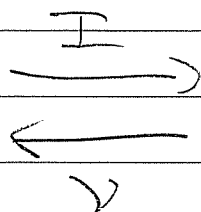
- (0)
- $(g(x))$ for irreducible $g(x) \in R[x]$
- $(p, f(x))$ where $p \in R$ is prime and $\bar{f}(x) \in R/(p)[x]$ is irreducible.

The third kind are the maximal ideals.

Corollary: $R[x]$ has Krull dim 2

(In general, $\dim(R[x]) \geq \dim(R) + 1$).

Remark: Galois connections



will NOT be on the exam.