5/28/14

Review of 661/662

Today	5/28	11-12
Fri	5/30	
Tue	6/3	
Thur	6/5	

Qualifying Exam Fri 6/6, 11-2pm

MTH 661 Topics:

- Abstract structure of groups
  - Isomorphism Theorems
  - Jordan-Hölder

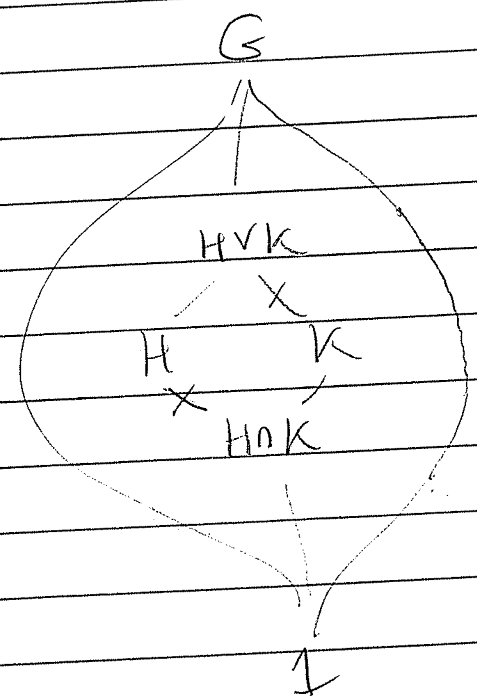
- Group Actions (G-sets)
  - The Class Equation
  - Cauchy's Theorem
  - Sylow Theory

- Matrix Groups
  - Theory of K-modules (Steinitz Exchange, Rank-Nullity, etc.)
  - GL and SL
    - elementary matrices, RREF
    - parabolic subgroups
    - over finite fields $K = \mathbb{F}_q$

- Representations of Groups
  - Definition of KG-modules
  - Properties of unitary matrices $U(n)$
  - Maschke's Theorem
    - indecomposable $\Rightarrow$ irreducible
    - if $|G| < \infty$ then every $\mathbb{C}G$-module is unitary
  - Schur's Lemma
    - Representations of abelian groups.

① Abstract Groups

The lattice of subgroups $\mathcal{L}(G)$.



If $H, K \leq G$ with $K \trianglelefteq G$ then

$$HK = \{hk : h \in H, k \in K\}$$

is a subgroup and $HK = H \vee K$. Moreover we have $H \cap K \trianglelefteq H$ and $K \trianglelefteq HK$ with

$$\frac{H}{H \cap K} \approx \frac{HK}{K}$$ Diamond Isomorphism

If H, K are finite then Lagrange says

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}$$

(Exercise: Prove this still holds when HK is not a group.)

If $G = HK$ and $H \cap K = 1$ then we say

$$G = H \ltimes K$$
"semi-direct product"

$\downarrow$

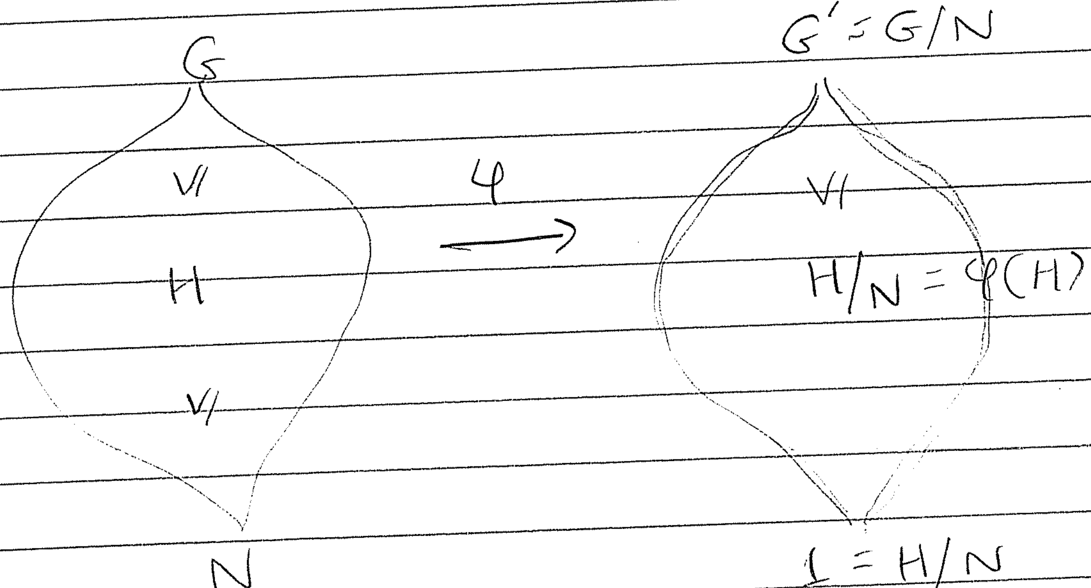
(Exercise: The semidirect product is direct $\iff H \trianglelefteq G \iff hk = kh$ for all $h \in H$, $k \in K$.)

Given a surjective group hom $\varphi : G \twoheadrightarrow G'$ with kernel N we have

$$G' \approx G/N$$

and furthermore we have an isomorphism of lattices

$$\mathcal{L}(G, N) \approx \mathcal{L}(G')$$



In addition to lattice structure, normality is also preserved:

$$H \trianglelefteq G \iff H/N \trianglelefteq G/N$$

in which case we also have

$$\frac{G/N}{H/N} \cong \frac{G}{H}.$$

We say a group is <u>simple</u> if it has no nontrivial normal subgroup.

Consider a chain of normal subgroups

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_r = 1.$$

"a sub-normal series"

Then G_i/G_{i+1} is <u>simple</u> $\iff$ there does <u>not</u> exist $G_i \triangleright N \triangleright G_{i+1}$.

We say the subnormal series is a "composition series" if G_i/G_{i+1} is simple $\forall\, i$, equivalently if the series is <u>maximal</u>.

Jordan-Hölder: If G has a comp. series then any two comp. series are equivalent in the sense that the set of simple quotients

$$\{G_i/G_{i+1}\}_i \qquad \underline{\text{the}} \text{ composition factors of } G$$

is the same.

Special Case of J-H: Unique prime factorization of integers.

Cyclic groups:

If $G = \langle g \rangle$ we say G is cyclic, in which case we have a surjective group hom

$$\varphi : (\mathbb{Z}, +) \twoheadrightarrow G$$
$$k \longmapsto g^k$$

Let $n\mathbb{Z}$ be the kernel. Then we have

$$G \approx \mathbb{Z}/n\mathbb{Z}.$$

FTCG: We have

$$\mathcal{L}(G) \approx \text{lattice of divisors of } n.$$

Proof: By correspondence we have

$$\mathcal{L}(G) \approx \mathcal{L}(\mathbb{Z}, n\mathbb{Z}).$$

E.g. $|\langle g \rangle| = 12$

$$\begin{tikzcd}
 & 1 \arrow[dl, dash, "3"'] \arrow[dr, dash, "2"] & & \\
3 \arrow[dr, dash, "2"'] & & 2 \arrow[dl, dash, "3"] \arrow[dr, dash, "2"] & \\
 & 6 \arrow[dr, dash, "2"'] & & 4 \arrow[dl, dash, "3"] \\
 & & 12 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \langle g \rangle \arrow[dl, dash, "\mathbb{Z}/3"'] \arrow[dr, dash, "\mathbb{Z}/2"] & & \\
\langle g^3 \rangle \arrow[dr, dash, "\mathbb{Z}/2"'] & & \langle g^2 \rangle \arrow[dl, dash, "\mathbb{Z}/3"] \arrow[dr, dash, "\mathbb{Z}/2"] & \\
 & \langle g^6 \rangle \arrow[dr, dash, "\mathbb{Z}/2"'] & & \langle g^4 \rangle \arrow[dl, dash, "\mathbb{Z}/3"] \\
 & & \langle g^{12} \rangle = 1 &
\end{tikzcd}$$

There are 3 composition series:

$3 \cdot 2 \cdot 2 \qquad \mathbb{Z}/3 \cdot \mathbb{Z}/2 \cdot \mathbb{Z}/2$

$2 \cdot 3 \cdot 2 \qquad \mathbb{Z}/2 \cdot \mathbb{Z}/3 \cdot \mathbb{Z}/2$

$2 \cdot 2 \cdot 3 \qquad \mathbb{Z}/2 \cdot \mathbb{Z}/2 \cdot \mathbb{Z}/3.$

Notation: We say G is <u>solvable</u> if its composition factors are abelian (i.e. $\mathbb{Z}/p$ for primes p)

Remark: Abelian groups are solvable.

<u>Exercise</u>: If $|G| = p^k$ with p prime, prove that G is solvable.
(I'll discuss this on Friday.)