Math 662 **Spring 2014**
Homework 2 **Drew Armstrong**

Problem 0. **(Drawing Pictures)** The equation $y^2 = x^3 - x$ defines a "curve" in the complex "plane" $\mathbb{C}^2$. What does it look like? Unfortunately we can only see real things, so we substitute $x = a + ib$ and $y = c + id$ with $a, b, c, d \in \mathbb{R}$. Equating real and imaginary parts then gives us **two simultaneous equations**:

$$a^3 - a - 3ab = c^2 - d^2, \tag{1}$$

$$b^3 + b - 3a^2b = -2cd. \tag{2}$$

These equations define a real 2-dimensional surface in real 4-dimensional space $\mathbb{R}^4 = \mathbb{C}^2$. Unfortunately we can only see 3-dimensional space so we will interpret the b coordiante as "time". Sketch the curve in real (a, c, d)-space at time $b = 0$. [Hint: It will look 1-dimensional to you.] Can you imagine what it looks like at other times b?

Problem 1. **(Local Rings)** Let R be a ring. We say R is local if it contains a unique (nontrivial) maximal ideal.

(a) Prove that R is local if and only if its set of non-units is an ideal.

(b) Given a prime ideal $P \leq R$, prove that the localization

$$R_P := \left\{ \frac{a}{b} : a, b \in R, b \notin P \right\}$$

is a local ring. [Hint: The maximal ideal is called PR_P.]

(c) Consider a prime ideal $P \leq R$. By part (b) we can define the residue field R_P/PR_P. Prove that we have an isomorphism of fields:

$$\text{Frac}(R/P) \approx R_P/PR_P.$$

[Hint: The most obvious map $R/P \to R_P/PR_P$ must factor through $\text{Frac}(R/P)$.]

Problem 2. **(Formal Power Series)** Let K be a field and consider the ring of formal power series:

$$K[[x]] := \left\{ a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots : a_i \in K \text{ for all } i \in \mathbb{N} \right\}.$$

The "degree" of a power series does not necessarily exist. However, for all nonzero $f(x) = \sum_k a^k x^k$ we can define the "order" $\text{ord}(f) :=$ the minimum k such that $a_k \neq 0$.

(a) Prove that $K[[x]]$ is a domain.

(b) Prove that $K[[x]]$ is a Euclidean domain with norm function $\text{ord} : K[[x]] - \{0\} \to \mathbb{N}$. (You can define $\text{ord}(0) = -\infty$ if you want.) [Hint: Given $f, g \in K[[x]]$ we have $f|g$ if and only if $\text{ord}(f) \leq \text{ord}(g)$, so the remainder is always zero.]

(c) Prove that the units of $K[[x]]$ are just the power series with nonzero constant term.

(d) Conclude that $K[[x]]$ is a local ring.

(e) Prove that $\text{Frac}(K[[x]])$ is isomorphic to the ring of formal Laurent series:

$$K((x)) := \left\{ a_{-n}x^{-n} + a_{-n+1}x^{-n+1} + a_{-n+2}x^{-n+2} + \cdots : a_i \in K \text{ for all } i \geq -n \right\}.$$

Problem 3. (Partial Fraction Expansion) To what extent can we "un-add" fractions? Let R be a PID. Consider $a, b \in R$ with $b = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where $p_1, \ldots, p_k$ are distinct primes and $e_1, \ldots, e_k \geq 1$.

(a) Prove that there exist $a_1, \ldots, a_k \in R$ such that

$$\frac{a}{b} = \frac{a_1}{p_1^{e_1}} + \frac{a_2}{p_2^{e_2}} + \cdots + \frac{a_k}{p_k^{e_1}}.$$

[Hint: First prove it when $b = pq$ with p, q coprime. Use Bézout.]

Now assume that R is a Euclidean domain with norm function $N : R - \{0\} \to \mathbb{N}$.

(b) Prove that there exist $c, r_{ij} \in R$ such that

$$\frac{a}{b} = c + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r_{ij}}{p_i^j},$$

where for all i, j we have either $r_{ij} = 0$ or $N(r_{ij}) < N(p_i)$. [Hint: If p is prime, prove that we can write $\frac{a}{p^e}$ as $\frac{q}{p^{e-1}} + \frac{r}{p^e}$ where either $r = 0$ or $N(r) < N(p)$. Then use (a).]

Now suppose that the norm function satisfies $N(a) \leq N(ab)$ and $N(a-b) \leq \max\{N(a), N(b)\}$ for all $a, b \in R - \{0\}$.

(c) Prove that the partial fraction expansion from part (b) is **unique**. [Hint: Suppose we have two expansions

$$c + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r_{ij}}{p_i^j} = \frac{a}{b} = c' + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{r'_{ij}}{p_i^j}.$$

Then we get a partial fraction expansion of zero:

$$\frac{0}{b} = \frac{a-a}{b} = (c - c') + \sum_{i=1}^{k} \sum_{j=1}^{e_i} \frac{(r_{ij} - r'_{ij})}{p_i^j}.$$

For all i, j define $\hat{b}_{ij} := b/p_i^j$, so that

$$b(c' - c) = \sum_{i=1}^{k} \sum_{j=1}^{e_i} (r_{ij} - r'_{ij}) \hat{b}_{ij}.$$

Suppose for contradiction that there exist i, j such that $r_{ij} \neq r'_{ij}$ and let j be maximal with this property. Use the last equation above to show that p_i divides $(r_{ij} - r'_{ij})$ and hence

$$N(p_i) \leq N(r_{ij} - r'_{ij}) \leq \max\{N(r_{ij}), N(r'_{ij})\} < N(p_i).$$

Contradiction.]

(d) If K is a field and $R = K[x]$ then the norm function $N(f) = \deg(f)$ satisfies the hypotheses of part (c) so the expansion is unique. **Compute** the unique expansion of

$$\frac{x^5 + x + 1}{(x+1)^2(x^2+1)} \in \mathbb{R}(x).$$

(e) If $R = \mathbb{Z}$ then the norm function $N(a) = |a|$ does **not** satisfy $|a - b| \leq \max\{|a|, |b|\}$. However, if we require remainders r, r' to be nonnegative then it is true that $|r - r'| \leq \max\{|r|, |r'|\}$ and the proof of uniqueness in (c) still goes through. **Compute** the unique expansion of $\frac{77}{12} \in \mathbb{Q}$ with nonnegative parameters $r_{ij} \geq 0$.