Tues Nov 19

HW 5 due Thurs Dec 5.
NO CLASS NEXT WEEK (Thanksgiving)

Where were we?

(A) Abstract Structure Theory of Groups ✓

(B) Matrix Groups ✓
Representations NEW TOPIC.

Q: What is a group?
A: A group is a collection of <u>symmetries</u> (of some <u>thing</u> X).

In the abstract definition of groups, the thing X is not mentioned.
So we have to put it back.

<u>Problem</u>: Given abstract group G, find a thing X and an injective homomorphism

$$\rho : G \hookrightarrow \mathrm{Aut}(X).$$

By Fundamental Hom Theorem, we have

$$G = G/_{\ker\rho} \approx \operatorname{im}\rho \leq \mathrm{Aut}(X),$$

so we have "represented" G as a group of symmetries of the thing X.

Definition: In general, given a thing X and a homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(X)$$

(not necessarily injective), we say that (ρ, X) is a representation of G.

Bonus Problem: Given abstract group G, find and classify all representations of G. ("Tannaka Duality": Under nice conditions we can recover G from its category of representations.)

First Case: X = a set.

Then a representation $\rho : G \to \mathrm{Aut}(X)$ is just an action $G \curvearrowright X$.

We also call (ρ, X) a G-set. Recall the Fundamental Theorem of G-sets:

Every G-set is a disjoint union of transitive G-sets (the orbits), and every transitive G-set X is isomorphic to G/H for some H. Furthermore, we have

$$G/H \approx G/K$$

if and only if $H = gKg^{-1}$ for some $g \in G$. Thus we have a bijection

$$\begin{array}{c} \text{Transitive} \\ G\text{-sets} \end{array} \longleftrightarrow \begin{array}{c} \text{Conjugacy classes} \\ \text{of subgroups of } G. \end{array}$$

///

Second Case: X = a vector space.

Let K be a field and consider $X = K^n$. Then

$$\mathrm{Aut}(X) = GL(n, K)$$

A homomorphism $\rho : G \to GL(n, K)$ is called a linear representation of G.

Our goal is to prove a Fundamental Theorem:

Let G be a finite group and let K be a field with $\mathrm{char}(K) \nmid |G|$. Then every linear representation $\rho : G \to GL(n, K)$ decomposes as a direct sum of irreducible representations. Furthermore, we have a bijection

$$\left\{ \begin{array}{c} \text{Irreducible} \\ \text{Representations} \\ \text{of } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Conjugacy classes of} \\ \text{elements of } G. \end{array} \right\}$$

In practice, we will assume $K = \mathbb{C}$.

The theory of linear representations developed very quickly

Dedekind $\rightsquigarrow$ Frobenius $\rightsquigarrow$ Done

letter: July 8, 1896 1901

BEGIN

Let G be a finite group and let K be a field with $\mathrm{char}(K) \nmid |G|$. (Think $\mathrm{char}(K) = 0$ if you like.)

Def: A linear representation of G is a group homomorphism

$$\rho : G \longrightarrow GL(V)$$

where V is a vector space over K. We say $\dim V$ is the degree of the representation. We will also call (ρ, V) a G-module.

Given G-modules (ρ, U) and (φ, V), a linear map $f : U \to V$ is called a morphism of G-modules (or a G-linear map) if $\forall g \in G$ the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle \varphi_g}\downarrow & & \downarrow{\scriptstyle \rho_g} \\ U & \xrightarrow[f]{} & V \end{array} \qquad \rho_g \circ f = f \circ \varphi_g$$

We say (ρ, U) and (φ, V) are isomorphic if $\exists$ G-linear isomorphism $f: U \to V$.
In this case we have $\dim U = \dim V = n$ (say).

If we fix coordinates on U, V then we get matrix representations

$$\rho: G \longrightarrow GL(n, K)$$
$$\varphi: G \longrightarrow GL(n, K)$$

and $f: U \to V$ can be thought of as the change-of-basis matrix $f \in GL(n, K)$.

Then to say that $\rho \cong \varphi$ means that $\forall g \in G$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\rho(g)"'] & V \arrow[d, "\varphi(g)"] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

$$\varphi(g) f = f \rho(g)$$
$$\varphi(g) = f \rho(g) f^{-1}$$

i.e. the matrices $\varphi(g)$ and $\rho(g)$ are simultaneously conjugte for all $g \in G$.

<u>Example</u>: Let S_3 = permutations of $\{1,2,3\}$ and consider the <u>defining</u> representation

$$\varphi : S_3 \to GL(3,\mathbb{C}), \quad \text{where}$$

$$\varphi(1) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \varphi((12)) = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$$

$$\varphi((13)) = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \quad \varphi((23)) = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

$$\varphi((123)) = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \quad \varphi((132)) = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

Observe: If the permutation $g \in S_3$ satisfies $g(i) = j$, then the matrix $\varphi(g) \in GL(3,\mathbb{C})$ satisfies

$$\varphi(g)\,\vec{e}_i = \vec{e}_j,$$

where $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

is the standard basis.

Let's choose a different basis

$$\vec{v}_1 = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$$
$$\vec{v}_2 = \vec{e}_1 - \vec{e}_2$$
$$\vec{v}_3 = \vec{e}_1 \phantom{- \vec{e}_2} - \vec{e}_3$$

The change of basis matrix is

$$[V \to \mathcal{E}] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

Changing $\mathcal{E} \to V$ gives an isomorphic matrix representation

$$\rho(g) = [\mathcal{E} \to V]\, \varphi(g)\, [V \to \mathcal{E}] \quad \forall g \in S_3$$

$$\rho(1) = \left(\begin{array}{c|cc} 1 & & \\ \hline & 1 & 0 \\ & 0 & 1 \end{array}\right), \quad \rho((12)) = \left(\begin{array}{c|cc} 1 & & \\ \hline & -1 & -1 \\ & 0 & 1 \end{array}\right)$$

$$\rho((13)) = \left(\begin{array}{c|cc} 1 & & \\ \hline & 1 & 0 \\ & -1 & -1 \end{array}\right), \quad \rho((23)) = \left(\begin{array}{c|cc} 1 & & \\ \hline & 0 & 1 \\ & 1 & 0 \end{array}\right)$$

$$\rho((123)) = \left(\begin{array}{c|cc} 1 & & \\ \hline & -1 & -1 \\ & 1 & 0 \end{array}\right), \quad \rho((132)) = \left(\begin{array}{c|cc} 1 & & \\ \hline & 0 & 1 \\ & -1 & -1 \end{array}\right)$$

Interesting: We say that $\rho : S_3 \to GL(3, \mathbb{C})$ is decomposable because we can write

$$\rho(g) = \left(\begin{array}{c|cc} A(g) & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{B(g)} \end{array}\right)$$

where $A : S_3 \to GL(1, \mathbb{C})$
$B : S_3 \to GL(2, \mathbb{C})$
are themselves representations. We write

$$\rho = A \oplus B.$$

(direct sum of representations)

We say a reresentation is indecomposable if it cannot be written as a direct sum.

Definition: Given a G-module (φ, V), we say subspace $U \leq V$ is a sub-G-module if

$$\forall g \in G, u \in U, \quad \varphi(g)u \in U.$$

"U is closed under the action of G".

If (φ, V) has no nontrivial sub-G-modules, we say it is an <u>irreducible</u> G-module.
("simple")

Clearly we have

$$\text{decomposable} \implies \text{reducible}$$
$$\text{indecomposable} \impliedby \text{irreducible}$$

But the other direction is not true in general.

☆ Theorem (Maschke, 1898):

If G is finite and $\text{char}(K) \nmid |G|$, then

$$\text{indecomposable} \implies \text{irreducible}$$

It follows that every f.d. G-module can be expressed as a direct sum of irreducibles.

("Prime factorization" of G-modules).

The problem of rep. theory is thus to classify the irreducible reps.

Thurs Nov 21

HW 5 due Thurs Dec 5

NO CLASS NEXT WEEK

Final Exam Thurs Dec 12 2:00 – 4:30 pm.

Given field K and finite group G, we define the group algebra

$$KG := \left\{ \sum_{g \in G} \alpha_g \cdot \bar{g} \ : \ \alpha_g \in K \right\}$$

This is <u>formal</u> linear combinations of group elements with associative product

$$\left(\sum_g \alpha_g \bar{g}\right)\left(\sum_h \beta_h \bar{h}\right) := \sum_{g,h} \alpha_g \beta_h \overline{gh}$$

It's a possibly noncommutative ring with identity $1_{KG} = 1_K \cdot \bar{1}_G$.

Now let M be a left KG-module, i.e. $\forall\, m, n \in M$, $r, s \in KG$ we have

- $1_{KG}\, m = m$
- $r(m+n) = rm + rn$
- $(r+s)m = rm + sm$
- $r(sm) = (rs)m$

If we restrict this action to the subfield $K \approx \{ k \cdot I_G : k \in K \} \subseteq KG$, we observe that M is a K-vector space.

Then for any $g \in G$ we define the map

$$\rho(g) : M \longrightarrow M$$
$$m \longmapsto 1_K \bar{g}\, m$$

Observe that $\rho(g) \in GL(M)$ and we have a homomorphism

$$\rho : G \longrightarrow GL(M).$$

Thus (ρ, M) is a representation (or a G-module). Conversely, consider any G-module

$$\rho : G \longrightarrow GL(M).$$

Then we can regard M as a KG-module by defining

$$\left(\sum \alpha_g \cdot \bar{g}\right) m := \left(\sum \alpha_g \rho(g)\right) m.$$

We obtain a natural bijection

Representations of G $\longleftrightarrow$ KG - modules

This allows us to use module theory to study representations.

Definition: We say $\rho : G \to GL(V)$ is decomposable if $\exists$ U, U' such that

$$V = U \oplus U'$$

as KG-modules.

Equivalently, $\exists$ a basis for V in which we can write

$$\rho(g) = \left(\begin{array}{c|c} \varphi(g) & 0 \\ \hline 0 & \mu(g) \end{array} \right)$$

where $\varphi : G \to GL(U)$ and $\mu : G \to GL(U')$.

We say $\rho : G \to GL(V)$ is reducible if $\exists$ nontrivial KG - submodule

$$0 \subsetneqq U \subsetneqq V$$

Equivalently, $\exists$ basis for V in which

$$\rho(g) = \left(\begin{array}{c|c} \varphi(g) & * \\ \hline 0 & * \end{array}\right)$$

where $\varphi : G \to GL(U)$.

Clearly we have

decomposable $\Rightarrow$ reducible
indecomposable $\Leftarrow$ irreducible

But the converse is not true in general.

☆ Theorem (Maschke, 1898):

If G is finite and $\mathrm{char}(K) \nmid |G|$, then

reducible $\Rightarrow$ decomposable

It follows that every f.d. KG-module is a direct sum of irreducibles.

("Prime Factorization").

The problem of rep. theory is thus to classify the irreducible representations.

Proof of Maschke (Abstract):

Let $0 \subsetneq U \subsetneq V$ be a nontrivial submodule. By extending a basis from U to V, $\exists$ subspace $U' \leq V$ such that

$$V = U \oplus U'$$

as vector spaces. But maybe U' is not stabilized by G. We will fix this.

Let $\pi : V \to U$ be the linear projection

$$\pi(u + u') := u.$$

Now define the "averaged" projection $\pi_G : V \to U$.

$$\pi_G(x) := \frac{1}{|G|} \sum_{g \in G} g\,\pi(g^{-1}x) \quad \forall x \in V.$$

- Note that π_G is defined because $\mathrm{char}(K) \nmid |G|$.
- Note that $\pi_G(x) \in U$ because U is G-stable
- Note that for all $u \in U$, $g \in G$ we have

$$g^{-1}u \in U \implies \pi(g^{-1}u) = g^{-1}u.$$

and hence

$$\pi_G(u) = \frac{1}{|G|} \sum_g \cancel{g} \cancel{g^{-1}} u = \frac{1}{|G|} \sum_g u$$

$$= \frac{1}{|G|} |G| u = u.$$

Furthermore, note that $\pi_G : V \to U$ is a morphism of KG-modules because $\forall h \in G$ and $x \in V$ we have

$$h\pi_G(x) = \left(\sum_g hg\pi(g^{-1}x)\right) / |G|.$$

$$= \left(\sum_g hg\pi(g^{-1}h^{-1}hx)\right) / |G|$$

$$= \left(\sum_k k\pi(k^{-1}hx)\right) / |G|$$

$$= \pi_G(hx)$$

Thus we have a split exact sequence of KG-modules

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, hook, "j"] & V \arrow[r, two heads] \arrow[l, bend left, "\pi_G"] & \ker \pi_G \arrow[r] & 0 \end{tikzcd}$$

where $j : U \hookrightarrow V$ is inclusion and $\pi_G \circ j = id_U$.
It follows from the Splitting Lemma (see HW 5.1) that

$$V = U \underset{\substack{\uparrow \\ \text{as } KG\text{-modules}}}{\oplus} \ker \pi_G$$

$\blacksquare$

That was quite abstract, so I'll present a more geometric proof for the case $K = \mathbb{C}$.

Recall the standard Hermitian form for $x, y \in \mathbb{C}^n$:

$$(x, y) = x^* y = \sum_i \overline{x_i} y_i$$

Consider $A \in GL(n, \mathbb{C})$ that preserves the form: $(Ax, Ay) = (x, y)$ $\forall\, x, y \in \mathbb{C}^n$.
Then we have

$$(Ax)^* Ay = x^* y$$
$$x^* A^* A y = x^* y \quad \forall\, x, y \in \mathbb{C}^n$$

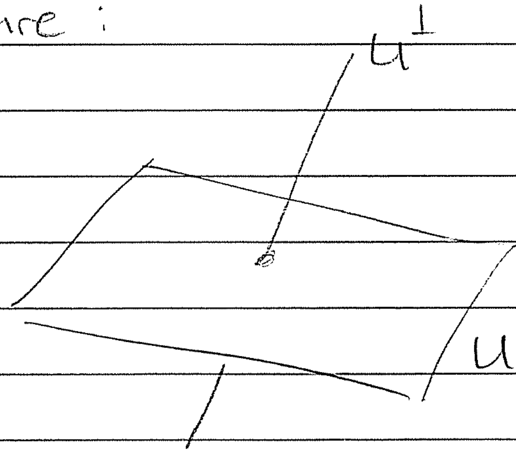
$$\Longrightarrow \quad A^* A = I.$$

We say that A is a <u>unitary</u> matrix.
(see HW 5.2)

<u>Lemma</u>: Let $A \in GL(n, \mathbb{C})$ be unitary ($A^*A = I$) and consider a subspace $\mathcal{U} \leq \mathbb{C}^n$ such that $A\mathcal{U} \subseteq \mathcal{U}$.
Then we also have $A^{-1}\mathcal{U}^\perp \subseteq \mathcal{U}^\perp$, where

$$\mathcal{U}^\perp = \{ x \in \mathbb{C}^n : (u, x) = 0 \ \ \forall\, u \in \mathcal{U} \}.$$

Picture:



If $A^*A = I$ then

A stabilizes $\mathcal{U}$

$\Updownarrow$

A^{-1} stabilizes $\mathcal{U}^\perp$.

Proof: Consider $x \in U^{\perp}$. Then $\forall\, u \in U$ we have

$$(u, A^{-1}x) = (Au, AA^{-1}x)$$
$$= (Au, x) = 0$$

because $Au \in U$. It follows that $A^{-1}x \in U^{\perp}$. ///

Proof of Maschke (Concrete):

Consider $\rho : G \to GL(n, \mathbb{C}^n)$. The matrices $\rho(g)$ may not be unitary, but we can define a new Hermitian form by averaging over the group:

$$(x, y)' := \frac{1}{|G|} \sum_{g \in G} (gx, gy) \quad \forall\, x, y \in \mathbb{C}^n.$$

Note that this form is G-invariant because $\forall\, h \in G$ and $x, y \in \mathbb{C}^n$ we have

$$(hx, hy)' = \left(\sum_g (hgx, hgy) \right) / |G|$$

$$= \left(\sum_k (kx, ky) \right) / |G|.$$

$$= (x, y)'.$$

With respect to the form $(\circ, \circ)'$ we now have

$$\rho : G \to U(n) = \{ A \in GL(n, \mathbb{C}) : A^* A = I \}.$$

(We say the representation is <u>unitary</u>).

If $(\rho, \mathbb{C}^n)$ has a nontrivial submodule

$$0 \subsetneq U \subsetneq \mathbb{C}^n$$

then we have $\mathbb{C}^n = U \oplus U^\perp$ as vector spaces. But since ρ is unitary it follows from the Lemma that $U^\perp$ is also G-stable.

Hence ρ is decomposable

$\blacksquare$

<u>Example</u>: The defining representation

$$\rho : S_3 \to GL(3, \mathbb{C})$$

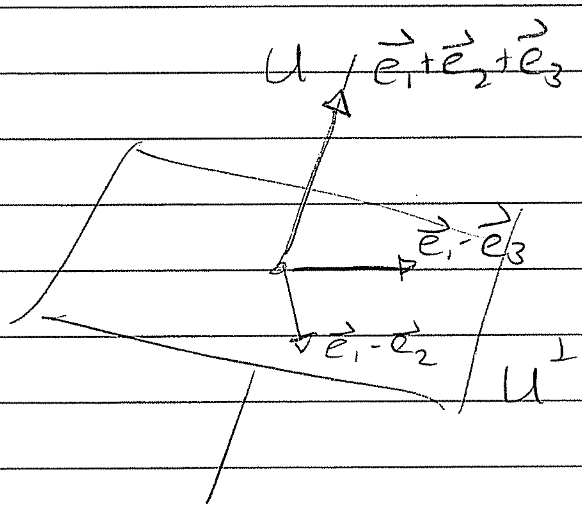
is unitary because permutation matrices are unitary. We know that ρ is reducible because it stabilizes the 1-dim subspace

$$U := \mathbb{C}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \leq \mathbb{C}^3$$

Hence it also stabilizes the orthogonal plane

$$U^\perp = \mathbb{C}(\vec{e}_1 - \vec{e}_2, \vec{e}_1 - \vec{e}_3)$$

and we get a decomposition



$$\mathbb{C}^3 = U \oplus U^\perp.$$

In this basis we have

$$\rho = \varphi \oplus \mu$$

where $\varphi(g) = (1) \;\; \forall\, g \in S_3$
"the trivial representation"

and

$$\mu(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu((12)) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\mu((13)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \mu((23)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mu((123)) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mu((132)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

I claim that μ is irreducible.

(Not immediately obvious.)

Tues Dec 3

HW 5 due Thurs
Review Session next Tues Dec 10
Final Exam Thurs Dec 12 2:00 – 4:30 pm.

This week: Rep Theory of Finite Groups.

Let G be a finite group and let K be a field with $\text{char}(K) \nmid |G|$.

Then (Maschke) every indecomposable KG-module is irreducible. Hence every f.g. KG-module is a direct sum of irreducibles.

"Prime Factorization"

Proof Idea: Let V be f.g. KG-module and assume V has an "inner product"

$$\langle \circ, \circ \rangle : V \times V \to K$$

(so assume $K = \mathbb{R}$ or $\mathbb{C}$) For all $A \in GL(V)$ we define the adjoint $A^* \in GL(V)$ by

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall\, x, y \in V.$$

Given any subspace $\mathcal{U} \leq V$ we get a direct sum decomposition

$$V = \mathcal{U} \oplus \mathcal{U}^{\perp}$$

where $\mathcal{U}^{\perp} = \{ x \in V : \langle u, x \rangle = 0 \ \forall u \in \mathcal{U} \}$.

Given $A \in GL(V)$ note that $A\mathcal{U} = \mathcal{U}$ $\iff A^* \mathcal{U}^{\perp} = \mathcal{U}^{\perp}$. Indeed, suppose $A\mathcal{U} = \mathcal{U}$ and consider $u \in \mathcal{U}$, $x \in \mathcal{U}^{\perp}$. Then $Au \in \mathcal{U}$, hence

$$0 = \langle Au, x \rangle = \langle u, A^* x \rangle.$$

Hence $A^* x \in \mathcal{U}^{\perp}$ ///

<u>Def</u>: We say $A \in GL(V)$ is an <u>isometry</u> if

$$\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in A.$$

This is equivalent to $A^* A = I$, i.e. $A^* = A^{-1}$.

Now let $0 \subsetneq U \subsetneq V$ be a KG-submodule, i.e. $\forall\, u \in U,\ g \in G$ we have $gu \in U$.

We have $V = U \oplus U^{\perp}$ as *spaces*, but $U^{\perp}$ might not be G-stable. So we define a *new* inner product

$$\langle x, y \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle gx, gy \rangle .$$

Note that every $g \in G$ is an isometry with respect to this form:

$$\langle gx, gy \rangle_G = \langle x, y \rangle \quad \forall\, x, y \in V$$

hence $g^* = g^{-1}$. Now consider the "G-orthogonal complement"

$$U^{\perp_G} := \{ x \in V : \langle u, x \rangle_G = 0 \ \forall\, u \in U \}.$$

Then we have $V = U \oplus U^{\perp_G}$ as spaces but note that $U^{\perp_G}$ is *also* G-stable. Indeed, since $g^* = g^{-1}$ $\forall\, g \in G$ we have

$$gU = U \iff g^{-1} U^{\perp_G} = U^{\perp_G} .$$

We conclude that

reducible $\Rightarrow$ decomposable

Thus the problem of rep theory is to classify the irreducible ("prime") KG-modules.

Here is the fundamental Lemma.

☆ Schur's Lemma: Let U, V be irreducible KG-modules and let $\varphi : U \to V$ be a KG-map. Then

$\varphi = 0$ OR φ is an isomorphism.

Proof: Since $\ker \varphi \leq U$ is a submodule and U is irreducible we have $\ker \varphi = U$ (in which case $\varphi = 0$) or $\ker \varphi = 0$ (in which case φ is injective). So assume $\ker \varphi = 0$. Then since $\operatorname{im} \varphi \leq V$ is a submod and V is irreducible we have $\operatorname{im} \varphi = 0$ (in which case $U = 0$) or $\operatorname{im} \varphi = V$

(in which case φ is surjective and hence bijective). ///

From now on we assume $K = \mathbb{C}$ (or at least algebraically closed)

☆ Schur's Lemma over $\mathbb{C}$:
Let U, V be irreducible $\mathbb{C}G$-modules and let $\varphi : U \to V$ be a $\mathbb{C}G$-map. Then

$$\varphi = 0 \quad \underline{\text{OR}} \quad \varphi = \lambda \, \mathrm{Id} \quad \text{for some } 0 \neq \lambda \in \mathbb{C}.$$

<u>Proof</u>: We know $\varphi = 0$ or φ is invertible. So assume φ is invertible and let $0 \neq \lambda \in \mathbb{C}$ be an eigenvalue (which exists by algebraic closure).

Then $\varphi - \lambda \mathrm{Id} : U \to V$ is a $\mathbb{C}G$-map that is <u>not</u> invertible, hence

$$\varphi - \lambda \mathrm{Id} = 0 \implies \varphi = \lambda \mathrm{Id}.$$

///

Application: Let A be an abelian group.

Then every irreducible $\mathbb{C}A$-module is 1-dimensional.

Proof: Let V be an irreducible $\mathbb{C}A$-mod. Then for all $a \in A$ the map $a : V \to V$ is a $\mathbb{C}A$-map because $\forall\, x \in V$, $b \in A$ we have

$$a(bx) = (ab)x = (ba)x = b(ax)$$

By Schur we conclude $a = \lambda \, Id : V \to V$. Since this is true for all $a \in A$ we see that every subspace $U \leq V$ is a submodule.

Since V is irreducible this implies that $\dim V = 1$. ///

Example: Let $G = \langle g \rangle$ be cyclic of order n, and let

$$\varphi : G \to GL(V) \text{ be irreducible}$$

Then $\dim V = 1$ and we can write

$$\varphi : G \longrightarrow GL(1, \mathbb{C}) = \mathbb{C}^\times$$

Note that φ is determined by the value $\varphi(g) \in \mathbb{C}^\times$ since $\varphi(g^k) = \varphi(g)^k \; \forall \, k$.

Note also that

$$\varphi(g)^n = \varphi(g^n) = \varphi(1) = 1$$

$\Longrightarrow$ $\varphi(g)$ is an nth root of 1.

We obtain a bijection

$\mathbb{C}$-irreps of $\mathbb{Z}/n$ $\longleftrightarrow$ $\mathbb{C}$ nth roots of 1.
$e^{2\pi i k/n}$.

Example: $\mathbb{Z}/4 = \{1, g, g^2, g^3\}$

elements

irreps	1	g	g^2	g^3
	1	1	1	1
	1	i	-1	$-i$
	1	-1	1	-1
	1	$-i$	-1	i

called a "character table"

Exercise: Find all the $\mathbb{C}$-irreps of the "Klein Four-Group" $\mathbb{Z}/2 \times \mathbb{Z}/2$

Example: Fourier Series

Consider the "circle group"

$$U(1) = \{ z \in \mathbb{C}^\times : \|z\| = 1 \}$$
$$= \{ e^{i\theta} : \theta \in \mathbb{R} \}$$

By Schur, every $\mathbb{C}$-irrep is 1-dimensional

$$\varphi : U(1) \to \mathbb{C}^\times.$$

Since $U(1)$ is compact, our proof of Maschke still holds, with

$$\langle x, y \rangle_G = \int_G \langle gx, gy \rangle \, dg.$$

Hence we can assume φ is unitary

$$\varphi : U(1) \to U(1).$$

Since φ is a homomorphism we have

$$\varphi\left(e^{i(\theta_1+\theta_2)}\right) = \varphi\left(e^{i\theta_1} e^{i\theta_2}\right)$$
$$= \varphi(e^{i\theta_1})\,\varphi(e^{i\theta_2}).$$

If we write $\varphi(\theta) := \varphi(e^{i\theta})$ for simplicity, this becomes

$$\varphi(\theta_1+\theta_2) = \varphi(\theta_1)\,\varphi(\theta_2)$$
$$\varphi(0) = 1.$$

Under mild hypotheses (e.g. Lebesgue-measurable) this implies that

$$\varphi(\theta) = e^{i\alpha\theta} \quad \text{for some } \alpha \in \mathbb{R}.$$

But since $e^{i\theta} = e^{i(\theta+2\pi k)}$ $\forall k \in \mathbb{Z}$, we also have

$$\varphi(\theta) = \varphi(\theta + 2\pi k)$$
$$e^{i\alpha\theta} = e^{i\alpha(\theta+2\pi k)}$$
$$\cancel{e^{i\alpha\theta}} = \cancel{e^{i\alpha\theta}}\, e^{i\alpha 2\pi k}$$
$$1 = e^{i\alpha 2\pi k} \quad \forall k \in \mathbb{Z}$$

$$\Longrightarrow \alpha k \in \mathbb{Z} \quad \forall k \in \mathbb{Z}$$
$$\Longrightarrow \alpha \in \mathbb{Z}.$$

We conclude that the $\mathbb{C}$-irreps of $U(1)$ are

$$\varphi_n(e^{i\theta}) = e^{in\theta}$$

for all $n \in \mathbb{Z}$. There exist topologies in which these form an orthonormal basis for the space of functions on the circle.

Expressing a function in this basis is called "Fourier Series".

Remark: This is true in general. The irreducible representations form a sort of basis for functions on a group. ("Harmonic Analysis")

~~The smallest nonabelian group is S_3, and it has a 2-dimensional irreducible representation~~

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Thurs Dec 5

HW 5 due now
Review Session next Tues
Final Exam next Thurs 2:00 – 4:30pm

Today: The end of rep. theory. (for us)

Recall Schur's Lemma over $\mathbb{C}$:

If U, V are irreducible $\mathbb{C}G$-modules and $\varphi : U \to V$ is a $\mathbb{C}G$-map then

$\varphi = 0$ OR $\varphi = \lambda \mathrm{Id}$ for some $\lambda \in \mathbb{C}$.

In other words:

$$\dim \mathrm{Hom}_G(U,V) = \begin{cases} 0 & \text{if } U \not\cong V \\ 1 & \text{if } U \cong V \end{cases}.$$

Today we will study $\dim \mathrm{Hom}_G(U,V)$ for general $\mathbb{C}G$-modules U, V (i.e. not necessarily irreducible).

Corollary of Schur:

Let G be abelian. Then every irreducible $\mathbb{C}G$-module is 1-dimensional.

We saw that the irreps of $\mathbb{Z}/n$ are precisely

$$\begin{aligned} \varphi_k : \mathbb{Z}/n &\longrightarrow \mathbb{C} \\ 1 &\longmapsto e^{2\pi i k/n} \end{aligned}$$

Thus there are n different irreps and we can collect them in the "character table" where $\omega = e^{2\pi i/n}$

	0	1	2	$\cdots$	d	$\cdots$	$n-1$
φ_0	1	1	1	$\cdots$	1	$\cdots$	1
φ_1	1	ω	ω^2	$\cdots$	ω^d	$\cdots$	ω^{n-1}
$\vdots$	$\vdots$						
φ_k	1	ω^k	ω^{2k}	$\cdots$	ω^{dk}	$\cdots$	$\omega^{(n-1)k}$
$\vdots$	$\vdots$						
φ_{n-1}	1	ω^{n-1}	$\omega^{2(n-1)}$	$\cdot$	$\cdot$	$\cdot$	$\omega^{(n-1)(n-1)}$

We saw that the irreps of $U(1)$ are precisely

$$\varphi_k : U(1) \longrightarrow U(1) \quad \text{for all } k \in \mathbb{Z}.$$
$$e^{i\theta} \longmapsto e^{ik\theta}$$

This leads to Fourier series.

Now consider the smallest nonabelian group

$S_3 = \{ 1, (12), (13), (23), (123), (132) \}$.

We know three representations of S_3.

	1	(12)	(13)	(23)	(123)	(132)
triv	1	1	1	1	1	1
ρ	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
sgn (=det)	1	-1	-1	-1	1	1.

I claim that triv, ρ, sgn are irreducible and they are the only irreps of S_3.

How can we prove this?

It's clear that triv, sgn are irred, but why is ρ irreducible?

The best way to study representations is through their "characters"

Def: Consider a $\mathbb{C}G$-module

$$\varphi : G \longrightarrow GL(V).$$

We define the character of φ to be

$$\chi_\varphi : G \longrightarrow \mathbb{C}.$$

$$g \longmapsto \operatorname{Trace}(\varphi(g)) = \sum \text{eigenvalues of } \varphi(g)$$

More generally, define a <u>class function</u> on G

$$\chi : G \to \mathbb{C}.$$

to satisfy $\chi(ghg^{-1}) = \chi(h)$ for all $g, h \in G$

<u>Theorem</u>: Characters are class functions.

<u>Proof</u>: Given $\varphi : G \to GL(V)$ we have

$$\begin{aligned}
\chi_\varphi(ghg^{-1}) &= \mathrm{Tr}\left(\varphi(ghg^{-1})\right) \\
&= \mathrm{Tr}\left(\varphi(g)\varphi(h)\varphi(g)^{-1}\right) \\
&= \mathrm{Tr}\left(\cancel{\varphi(g)^{-1}\varphi(g)}\,\varphi(h)\right) \\
&= \mathrm{Tr}\left(\varphi(h)\right) \\
&= \chi_\varphi(h).
\end{aligned}$$

$\blacksquare$

[Recall: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.]

Given a conjugacy class $\mathcal{C} \subseteq G$
we define the indicator class function

$$\chi_{\mathcal{C}}(g) := \begin{cases} 1 & g \in \mathcal{C} \\ 0 & g \notin \mathcal{C} \end{cases}$$

Easy Theorem: Class functions $G \to \mathbb{C}$ form a $\mathbb{C}$-module by defining

$$(\chi + \mu)(g) = \chi(g) + \mu(g)$$
$$(\alpha\chi)(g) = \alpha(\chi(g))$$

$\forall\, g \in G,\ \alpha \in \mathbb{C}$. Furthermore, the indicator functions χ_C are a basis. Hence

dim (class func.) = # conj. classes.

Let $\mathbb{C}[G]^G$ denote the space of class functions.

☆ Fundamental Theorem ☆

Let G be finite. Then the characters of irreducible $\mathbb{C}G$-modules are a basis for $\mathbb{C}[G]^G$.

Proof Sketch:

First we define an inner product on $\mathbb{C}[G]^G$.

Given $\chi, \mu \in \mathbb{C}[G]^G$, let

$$\langle \chi, \mu \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \mu(g).$$

[You know this is an inner product.]

Next, given any $\mathbb{C}G$-modules U, V with characters χ_U, χ_V, we have

$$\langle \chi_U, \chi_V \rangle = \dim \operatorname{Hom}_G(U, V).$$

[Proof omitted]

By Schur's Lemma this shows that the irred. characters are orthonormal. Indeed, given irred. U, V we have

$$\langle \chi_U, \chi_V \rangle = \dim \operatorname{Hom}_G(U, V) = \begin{cases} 1 & \chi_U = \chi_V \\ 0 & \chi_U \neq \chi_V \end{cases}$$

In particular, they are independent. Then we show that they span.

[Proof omitted] ///

Corollary:

$\#$ irred. $\mathbb{C}G$-modules $=$ $\#$ conj. classes of G.

Corollary: Every $\mathbb{C}G$-module is determined by its character.

Proof: Let $U_1, \ldots, U_k$ be all the irred. $\mathbb{C}G$-modules with chars $\chi_1, \ldots, \chi_k$. Given any $\mathbb{C}G$-mod. V, decompose into irreds by Maschke.

$$V = a_1 U_1 \oplus a_2 U_2 \oplus \cdots \oplus a_k U_k.$$

Taking traces gives

$$\chi_V = a_1 \chi_1 + a_2 \chi_2 + \cdots + a_k \chi_k.$$

Finally, we have

$$\langle \chi_j, \chi_V \rangle = \sum_i a_i \langle \chi_j, \chi_i \rangle = a_j$$

Thus the a_j and hence V are determined by the character χ_V.

///

Finally, we consider the regular representation of G. This is just $\mathbb{C}G$ considered as a module over itself.

Note that $g \in G$ acts on $\mathbb{C}G$ by permuting the basis vectors e_g !

$$g \cdot e_h = e_{gh}.$$

Thus g is a $|G| \times |G|$ permutation matrix and the trace is

$$\chi_{\mathbb{C}G}(g) = \#\{h \in G : gh = h\}$$

$$= \begin{cases} |G| & g = 1 \\ 0 & g \neq 0. \end{cases}$$

Thus the coefficient of irrep. χ_j in $\chi_{\mathbb{C}G}$ is

$$\langle \chi_j, \chi_{\mathbb{C}G} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_j(g)}\, \chi_{\mathbb{C}G}(g).$$

$$= \frac{1}{\cancel{|G|}} \overline{\chi_j(1)} \cdot \cancel{|G|}$$

$$= \overline{\chi_j(1)} = \dim V_j.$$

In summary, we have

$$\chi_{\mathbb{C}G} = \sum_i (\dim V_i)\, \chi_i$$

(Each irrep. occurs in $\mathbb{C}G$ with multiplicity equal to its dimension.).

Example: S_3

The three irreps have characters

	1	(12)	(13)	(23)	(123)	(132)
χ_{triv}	1	1	1	1	1	1.
χ_ρ	2	0	0	0	-1	-1
χ_{sgn}	1	-1	-1	-1	1	1

Note that

$$\langle \chi_\rho, \chi_\rho \rangle = \frac{1}{6}\left[2^2 + 0^2 + 0^2 + 0^2 + (-1)^2 + (-1)^2 \right]$$

$$= \frac{1}{6}\left[4 + 1 + 1 \right] = 1.$$

This implies that χ_ρ is irreducible.

[In general, if $\chi = \sum a_i \chi_i$, then

$$\langle \chi, \chi \rangle = a_1^2 + a_2^2 + \cdots + a_k^2.$$

Hence $\langle \chi, \chi \rangle = 1 \iff \chi = \chi_i$ for some i.]

Finally consider the regular character

$$\chi_{reg} \quad 6 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.$$

Note that

$$\chi_{triv} + 2\chi_\rho + \chi_{sgn}$$

$$\begin{array}{rcccccc} = & 1 & 1 & 1 & 1 & 1 & 1 \\ + & 4 & 0 & 0 & 0 & -2 & -2 \\ + & 1 & -1 & -1 & -1 & 1 & 1 \\ \hline & 6 & 0 & 0 & 0 & 0 & 0 \end{array}$$

As Expected.

Epilogue: Fourier Again.

For an abelian group G, the space of class functions is just the space of functions $\mathbb{C}[G]$, i.e. it is the regular representation.

Thus every function $f: G \to \mathbb{C}$ can be expressed uniquely as a linear combination of irred. characters.

When $G = U(1)$ this says

" Every function $f: U(1) \to \mathbb{C}$ from the circle to $\mathbb{C}$ can be expressed uniquely as a linear combination of the characters $\chi_n(\theta) = e^{in\theta}$:

$$f(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}.$$

Fourier coefficients "

WARINING: $\exists$ topological and analytic subtleties.