1/22/15

HW 1 : TBA soon
Quiz 1 : Tuesday in class.

The quiz will cover material from the first three lectures (lecture notes are on the webpage).

The first 3 lectures were "Introduction".

Summary:

- P.O. : Two strings (with common T and μ) will sound good together if their lengths have the ratio of small whole numbers.

- Two most consonant ratios are

$$2/1 \quad \text{"octave"}$$
$$3/2 \quad \text{"perfect fifth"}$$

The equation

$$m \text{ fifths} = n \text{ octaves}$$

is impossible, but it has an approximate solution

12 fifths $\approx$ 7 octaves

$$\left(\frac{3}{2}\right)^{12} \approx \left(\frac{2}{1}\right)^{7}$$

This is why we divide the octave into twelve intervals

- V.O.: In a string with tension T and density μ, any disturbance will propagate with velocity

$$\sqrt{T/\mu}\ .$$

If the string has length l, then any disturbance will repeat with frequency

$$f = \frac{1}{2l}\sqrt{\frac{T}{\mu}}\ .$$

Disturbances with symmetry can repeat at frequencies $f, 2f, 3f, 4f, \ldots$

- Helmholtz' Observation, 1877 (H.O.):

The dissonance of two tones played together is caused by "beats" between the higher partials.

- Given two strings of lengths l_1, l_2 with fundamental frequencies f_1, f_2 and equal T, μ, we have

$$\frac{l_1}{l_2} = \frac{f_2}{f_1}$$

If $f_2/f_1 = a/b$ then the partials are never closer than f_1/b. If b is small this means f_1/b is large, so "beating" is minimized.

- Putting everything together, we have a "modern" explanation of P.O.

BEGIN THE COURSE.

Our first goal is to understand the mathematics of beats. It is based on the trigonometric identity

$$\sin(u) + \sin(v) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right).$$

There are two steps:

1. Explain why trigonometry has anything to do with sound.
2. Study trigonometry.

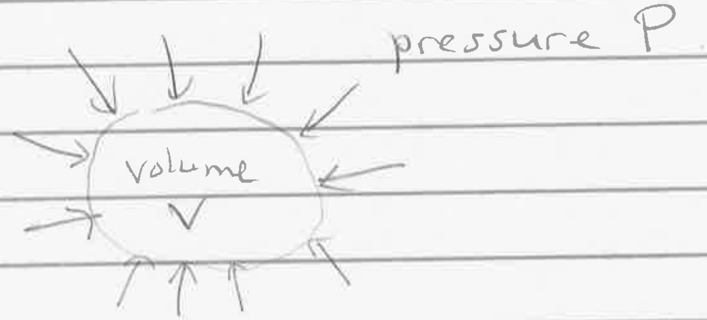
1. Why Trigonometry?

Sound is a (longitudinal) pressure wave in air. It is mathematically similar to (transverse) waves on a stretched string. For example, the speed of sound in air is

$$c = \sqrt{\frac{K}{\rho}}.$$

Here, K is the "bulk modulus" of air. It is thought of as resistance to a compressive force.

Specifically, as pressure is applied to a contained gas the volume goes down:



Thus the slope $\frac{dP}{dV}$ is negative. For a volume V of air under pressure P, the air will resist contraction by exerting an outward pressure

$$K := -V \frac{dP}{dV}.$$

Recall that pressure is defined as

$$\text{pressure} = \frac{\text{force}}{\text{area}} \sim \frac{kg \cdot m/s^2}{m^2} = \frac{kg}{m s^2}$$

Thus K is measured in units

$$K \sim \frac{kg}{m s^2}.$$

Since density is

$$\rho = \frac{\text{mass}}{\text{volume}} \sim \frac{kg}{m^3}$$

we have

$$\frac{K}{\rho} \sim \frac{kg/ms^2}{kg/m^3} = \frac{m^2}{s^2} = \left(\frac{m}{s}\right)^2$$

as expected. The formula for the speed of sound,

$$c = \sqrt{\frac{K}{\rho}}$$

is called the Newton-Laplace equation. Newton mentioned this idea in his Principia Mathematica (1687) and the details were later corrected by Laplace.

Q: So where does trigonometry come in?

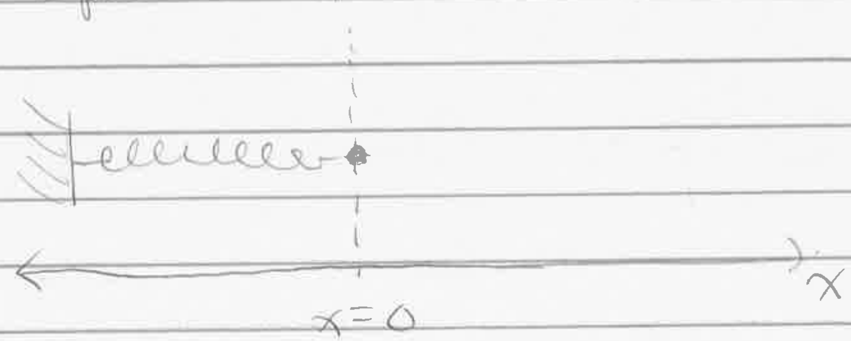
A: We model waves with trigonometric functions because they all satisfy the same differential equation.

☆ Hooke's Law:

The force needed to compress or extend a spring is directly proportional to the displacement:

$$\text{force} = -k^2 \cdot \text{displacement}$$

Here k is a constant describing the "stiffness" of the spring. Since $-k^2 < 0$, the spring force resists the displacement.



Let $x(t)$ be the displacement at time t.

Then Hooke's Law and Newton's 2nd say

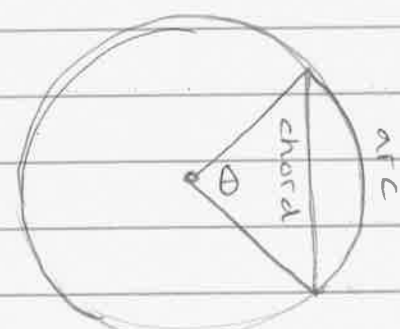
$$\frac{d^2x}{dt^2} = -k^2 \cdot x(t).$$

We will see that the general solution is

$$x(t) = A\sin(kt) + B\cos(kt).$$

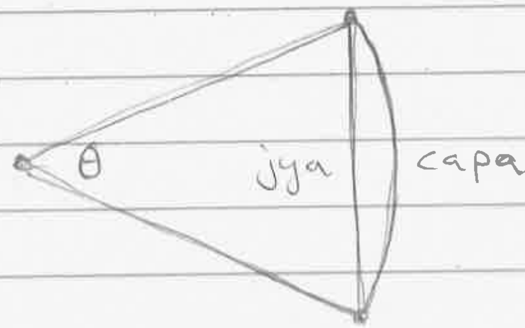
But first we need to define sin & cos.

Trigonometry was first used by Ptolemy in his study of astronomy. In the "Almagest" (~AD 150) he gave a table of chords.



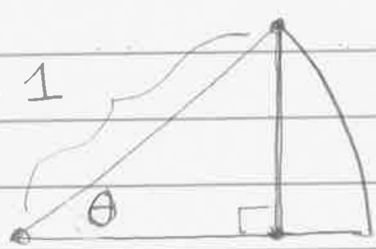
The length of the chord is a function of the arc/angle θ.

This was imported into Indian mathematics and they changed the name



"jya" = "chord"
"samastajya" = "bowstring" } Sanskrit
"capa" = "bow"

At some point the chord was replaced by the half-chord ("jya-ardha"). This appears first in the "Aryabhatiya" (AD 499).



We are only interested in the ratio of jya-ardha to θ so we might as well assume the radius is 1.

Then jya-ardha $= \sin(\theta)$.

From "jya-ardha" to "sine":

jya = chord	Sanskrit
jiba (nonsense)	phonetic Arabic
jyb	omit vowels
(jaib = bosom)	Arabic
sinus = bosom	Latin
sine	English

Believe it or not!

For us the most important fact about trigonometry is contained in the following identities:

$$\left\{\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta. \end{aligned}\right.$$

What does this look like to you?

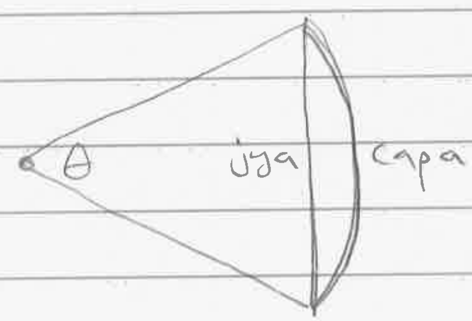
1/27/15

HW1 : TBA Thursday.
Today : Quiz 1 (20 minutes)

Last time I asked the questions:

- Why and how is trigonometry used in the study of music?

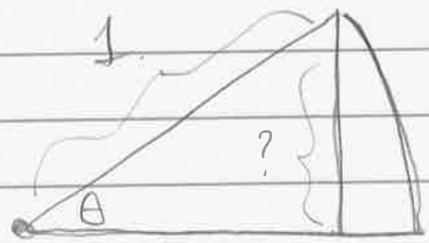
We saw how trigonometry began as a stretched bow in Classical India:



jya = bowstring (or chord of circle)
capa = bow (or arc of circle)

They were interested in computing the ratio of the chord to the arc. For this purpose the radius is arbitrary, so we take it to be 1.

At some point they decided to cut the chord in half:

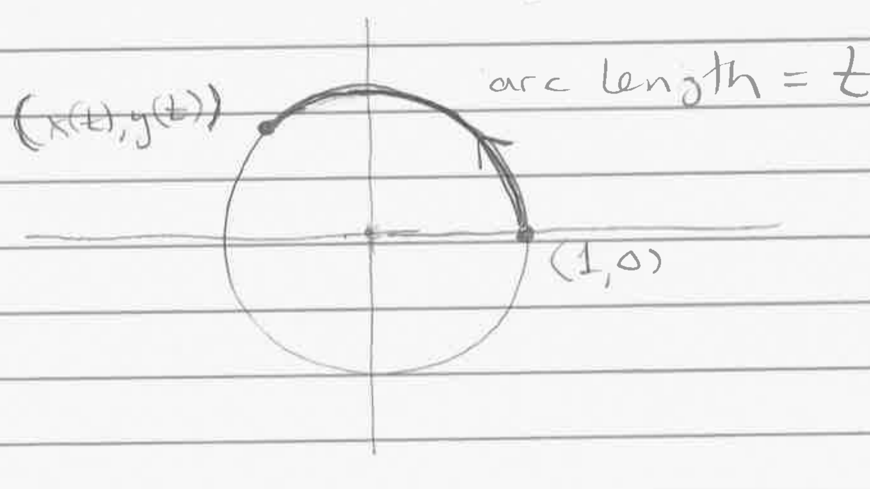


jya-ardha = half-chord.

In our language, jya-ardha = $\sin(\theta)$.

These days we define trigonometry differently.

Definition: Let $x(t)$ and $y(t)$ be the functions that parametrize the unit circle by arc length, starting at $(1,0)$ and traveling counterclockwise.



Then we define

$$\cos(t) := x(t)$$
$$\sin(t) := y(t)$$

["Trigonometry" means "triangle measurement", but it should really be "circle measurement".]

Observations:

- $(x(t), y(t))$ is on the circle $x^2+y^2=1$, hence

$$\cos^2(t) + \sin^2(t) = 1.$$

"Abuse of Notation"

- The perimeter of the unit circle is 2π and hence

$$\cos(t+2\pi) = \cos(t)$$
$$\sin(t+2\pi) = \sin(t).$$

"Periodic with period 2π."

The most important fact about trigonometry is the following.

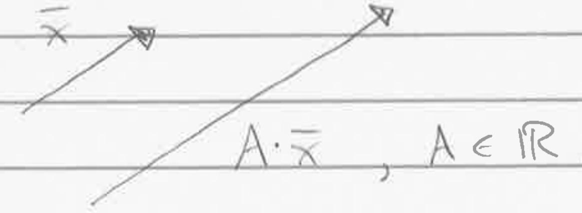
☆ Theorem: For all $\alpha, \beta \in \mathbb{R}$ we have

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta .$$

///

There are many ways to prove this, but there is only <u>one</u> correct way to do it. It involves linear algebra, which is just as well because we need linear algebra soon anyway.

Recall that vectors in the Cartesian plane $\mathbb{R}^2$ can be added and scaled



We say that a function $f: \mathbb{R}^2 \to \mathbb{R}^2$ is <u>linear</u> if it preserves these operations:

$$f(A \cdot \bar{x} + B \cdot \bar{y}) = A \cdot f(\bar{x}) + B f(\bar{y}).$$

Equivalently, we can think of a linear function $f: \mathbb{R}^2 \to \mathbb{R}^2$ as a "2×2 matrix".

Here's How:

Suppose $f: \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Recall that $\mathbb{R}^2$ has <u>standard</u> <u>basis</u> <u>vectors</u>

$$\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Every other vector $\bar{x} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$

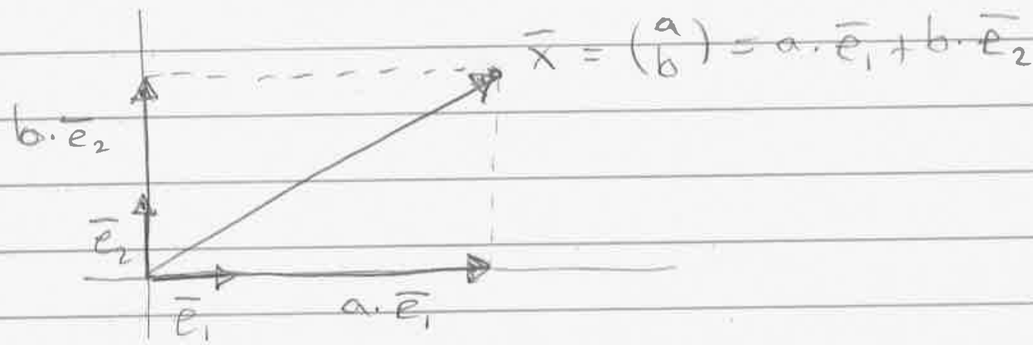
has a <u>unique</u> expression in terms of these.

i) Algebraically, $\bar{x} =$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= a \cdot \bar{e}_1 + b \cdot \bar{e}_2 \quad ///$$

ii) Geometrically,



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Now suppose we know what f does to the basis vectors, say

$$f(\bar{e}_1) = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } f(\bar{e}_2) = \begin{pmatrix} c \\ d \end{pmatrix}$$

Then f applied to a general vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is

$$f\begin{pmatrix} x \\ y \end{pmatrix} = f(x\cdot\bar{e}_1 + y\cdot\bar{e}_2)$$

$$= x\cdot f(\bar{e}_1) + y\cdot f(\bar{e}_2)$$

$$= x\cdot\begin{pmatrix} a \\ b \end{pmatrix} + y\cdot\begin{pmatrix} c \\ d \end{pmatrix}$$

$$= \begin{pmatrix} xa \\ xb \end{pmatrix} + \begin{pmatrix} yc \\ yd \end{pmatrix} = \begin{pmatrix} xa+yc \\ xb+yd \end{pmatrix}.$$

Gee, it would be nice to have a compact notation for that calculation.

Definition: We set

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$$

This allows us to identify the function f with a 2×2 matrix,

$$[f] := \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

In general I put square brackets around something when I want to write it "in coordinates". This gives us a cute notation:

$$[f(\bar{x})] = [f] \cdot [\bar{x}]$$

apply function f to vector $\bar{x}$, then write in coordinates

multiply matrix $[f]$ by column vector $[\bar{x}]$.

So linear functions $\mathbb{R}^2 \to \mathbb{R}^2$ are the same as 2×2 matrices.

Q: Linear functions $\mathbb{R}^2 \to \mathbb{R}^2$ can be <u>composed</u>:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R}^2 \end{tikzcd}$$

How does this work in coordinates?

$$[f \circ g] = ?$$

A: The notation suggests that

$$[f \circ g] = [f] \cdot [g]$$

is some kind of "multiplication" of 2×2 matrices. The answer is predetermined; we just need to do a computation to work out the details.

I'll leave the computation to you (HW 1).

<u>Theorem</u>: Consider linear functions f, g with matrices

$$[f] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad [g] = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}$$

Then the composition function has matrix

$$[f \circ g] = \begin{pmatrix} aa' + cb' & ac' + cd' \\ ba' + db' & bc' + dd' \end{pmatrix}$$

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What does this have to do with trigonometry?

1/29/15

HW 1 due Tues Feb 10

Right now we are trying to prove the most important fact about trigonometry:

$$\left.\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned}\right\}.$$

The correct way to do this is via linear algebra.

Recall: We say a function $f: \mathbb{R}^2 \to \mathbb{R}^2$ is <u>Linear</u> if it satisfies

$$f(A\bar{x} + B\bar{y}) = Af(\bar{x}) + Bf(\bar{y})$$

for all vectors $\bar{x}, \bar{y} \in \mathbb{R}^2$ and scalars $A, B \in \mathbb{R}$.

Let $\bar{e}_1, \bar{e}_2 \in \mathbb{R}^2$ be the standard basis. If $\bar{x} = a\bar{e}_1 + b\bar{e}_2$ then we write

$$[\bar{x}] = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{"in standard coordinates"}.$$

In particular, we have

$$\bar{e}_1 = 1\bar{e}_1 + 0 \cdot \bar{e}_2 \implies [\bar{e}_1] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\bar{e}_2 = 0\bar{e}_1 + 1\bar{e}_2 \implies [\bar{e}_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If $f: \mathbb{R}^2 \to \mathbb{R}^2$ is <u>linear</u> with

$$[f(\bar{e}_1)] = \begin{pmatrix} a \\ b \end{pmatrix} \quad \& \quad [f(\bar{e}_2)] = \begin{pmatrix} c \\ d \end{pmatrix}$$

then we write

$$[f] = \Big([f(\bar{e}_1)] \; [f(\bar{e}_2)] \Big) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

We define the "product" of a 2×2 matrix and a 2×1 column so that the following very desirable notation makes sense:

$$[f(\bar{x})] = [f][\bar{x}]$$

This is what it is.

Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be linear functions.
We define the "product" of 2×2 matrices so that the following very desirable notation makes sense:

$$[f \circ g] = [f][g]$$

This is what it is.

You will show on HW1 that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix} = \begin{pmatrix} aa' + cb' & ac' + cd' \\ ba' + db' & bc' + dd' \end{pmatrix}$$

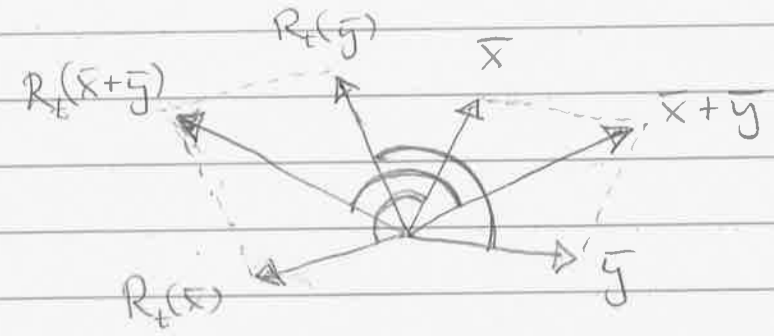
///

So What?

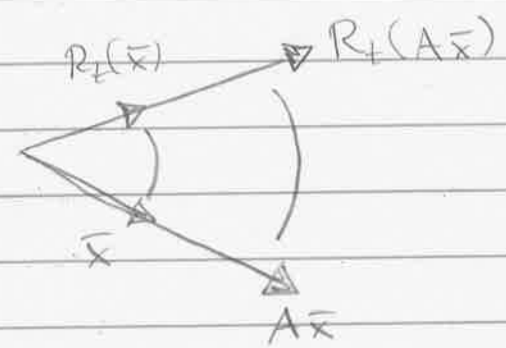
The reason we care is because <u>rotation</u> is a linear function.

Let $R_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates around the origin counterclockwise by angle t. Is this linear?

Yes! Note that $R_t(\bar{x}+\bar{y}) = R_t(\bar{x}) + R_t(\bar{y})$:



Note that $R_t(A\bar{x}) = A\,R_t(\bar{x})$:



Thus R_t can be expressed as a matrix.

$$[R_t] = ?$$

By <u>definition</u> of $\sin t$ and $\cos t$, we have

$$[R_t(\bar{e}_1)] = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix},$$

and by an easy symmetry we have

$$[R_t(\bar{e}_2)] = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

We conclude that

$$[R_t] = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

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Finally we can prove the most important fact about trigonometry.

☆ Theorem: For all $\alpha, \beta \in \mathbb{R}$ we have

$$\left.\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}\right\}$$

<u>Proof</u>: Rotation by β <u>followed</u> by rotation by α is the same as rotation by $\alpha+\beta$, i.e.,

$$R_{\alpha+\beta} = R_\alpha \circ R_\beta.$$

$\downarrow$

Expressing this in coordinates gives

$$[R_{\alpha+\beta}] = [R_\alpha \circ R_\beta]$$
$$= [R_\alpha][R_\beta].$$

Explicitly,

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}.$$

Comparing entries gives the result. $\blacksquare$

Remark: The set of rotations of $\mathbb{R}^2$ forms a "group" under composition.

$$R_\alpha \circ R_\beta = R_{\alpha+\beta}$$

The famous name for this group is $SO(2)$, the "special orthogonal" group.

Every element of $SO(2)$ is a linear combination of two special matrices:

$$[R_t] = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$= \cos t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$= \cos t \, [R_0] + \sin t \, [R_{\pi/2}].$$

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the identity matrix. It corresponds to the identity function (the "do nothing" function).

- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the "rotate $90°$" function. It has a special algebraic property:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= -1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the square root of the negative of the identity.

We can simplify the notation if we simply write

$$\underline{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \underline{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that $\underline{i}^2 = -\underline{1}$. Then our theorem says

$$(\cos\alpha \cdot \underline{1} + \sin\alpha \cdot \underline{i})(\cos\beta \cdot \underline{1} + \sin\beta \cdot \underline{i})$$

$$= \cos(\alpha+\beta) \cdot \underline{1} + \sin(\alpha+\beta) \cdot \underline{i}$$

In particular, we have

$$(\cos t \cdot \underline{1} + \sin t \cdot \underline{i})^n = \cos(nt) \cdot \underline{1} + \sin(nt) \cdot \underline{i}$$

This was discovered in 1707 by Abraham de Moivre. In 1749, Euler took this a step further by considering the function

$$f(\alpha) := \cos\alpha \cdot \underline{1} + \sin\alpha \cdot \underline{i}$$

He realized that the property

$$f(\alpha)\cdot f(\beta) = f(\alpha+\beta)$$

implies that $f(t) = e^{ct}$ for some constant t. What is the constant?

Well, $f'(t) = c\,e^{ct}$, so $f'(0) = c$.
On the other hand,

$$\begin{aligned} f'(t) &= -\sin t\cdot \underline{1} + \cos t\cdot \underline{i} \\ f'(0) &= -\cancel{\sin 0}\cdot \underline{1} + \cos 0\cdot \underline{i} \\ &= \underline{i} \end{aligned}$$

He concluded that

$$\cos t\cdot \underline{1} + \sin t\cdot \underline{i} = e^{\underline{i}t} = e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}}$$

Does it really make sense to raise e to the power of a matrix?

Sure. Why not?

2/3/15

HW 1 due Tues Feb 10

Last time we developed "Euler's Formula".

If we believe in imaginary numbers (e.g., $i^2 = -1$), it can be written as

$$e^{it} = \cos t + i \sin t .$$

If we don't believe in imaginary numbers, it can be written in terms of 2×2 matrices with real entries

$$e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} = \cos t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

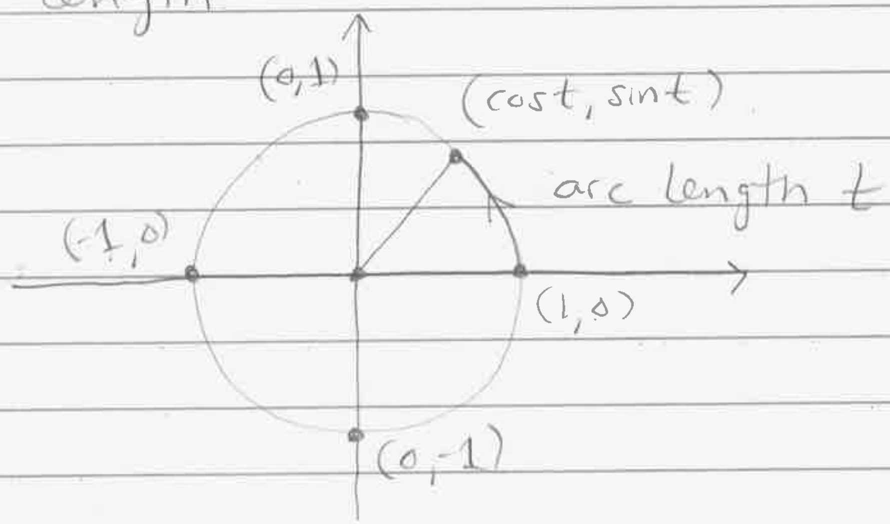
$$e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

The proof can be phrased as follows:

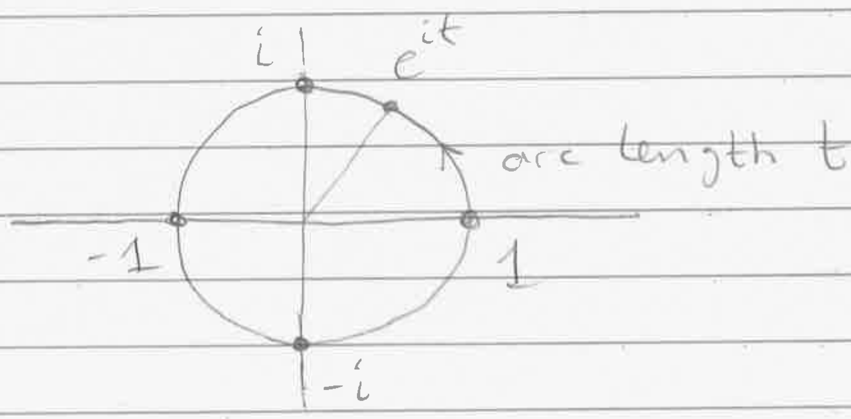
"Rotation by β followed rotation by α is the same as rotation by $\alpha + \beta$"

The rest is details.

Euler's formula gives us a new language for trigonometry. Recall that we defined $\cos t$ and $\sin t$ as the functions that parametrize the unit circle by arc length



If we reinterpret $\mathbb{R}^2$ as $\mathbb{C}^1$ then the unit circle is parametrized by e^{it}.



This often simplifies notation.

We can use Euler's formula to get back to the "angle sum identities":

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = e^{i(\alpha+\beta)}$$

$$= e^{i\alpha + i\beta} = e^{i\alpha} \cdot e^{i\beta}$$

$$= (\cos\alpha + i\sin\alpha) \cdot (\cos\beta + i\sin\beta)$$

$$= \cos\alpha\cos\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta + \cancel{i}^{2}\sin\alpha\sin\beta$$

$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).$$

Comparing real and imaginary parts gives

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Corollary: For all real numbers $u, v \in \mathbb{R}$, we have

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right).$$

Proof: Note that

$$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

[because $\cos(-\beta) = \cos\beta$ & $\sin(-\beta) = -\sin\beta$].

Then adding $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$ gives

$$\begin{array}{r} \sin\alpha\cos\beta + \cancel{\cos\alpha\sin\beta} \\ + \ \underline{\sin\alpha\cos\beta - \cancel{\cos\alpha\sin\beta}} \\ 2\sin\alpha\cos\beta + 0. \end{array}$$

Hence

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta.$$

Letting $\begin{array}{l} u = \alpha+\beta \\ v = \alpha-\beta \end{array}$ $\left(\begin{array}{l} \alpha = (u+v)/2 \\ \beta = (u-v)/2 \end{array}\right)$

gives the desired formula. ∎

Finally we can explain the mathematics of beats.

Suppose we listen to two pure tones played at frequencies 440 Hz and 442 Hz. We assume that the sounds are modeled by pure sine waves

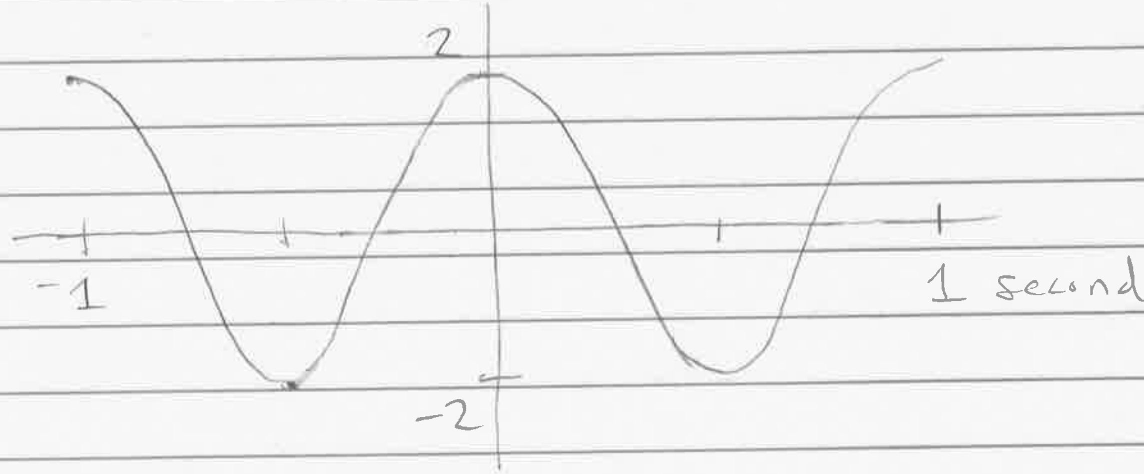
$$\sin(440 \cdot 2\pi t) \quad \& \quad \sin(442 \cdot 2\pi t).$$

The superposition of these two waves is

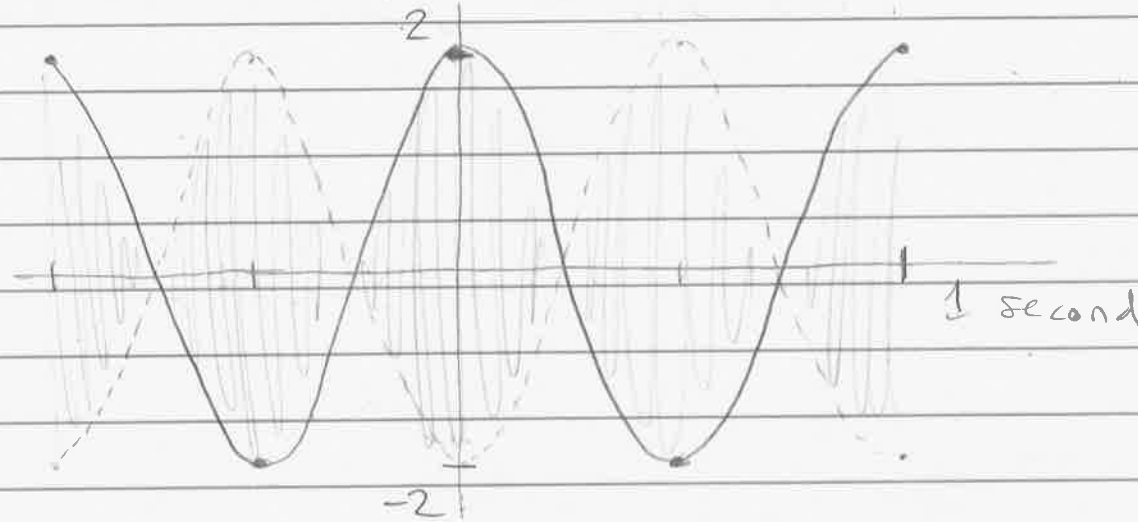
$$2\sin(441 \cdot 2\pi t)\cos(2\pi t).$$

What does this look/sound like?

The graph of $2 \cdot \cos(2\pi t)$ looks like



Multiplying this by $\sin(441 \cdot 2\pi t)$ looks something like



This sounds like a note with pitch 441 Hz turning on and off twice per second

In general, adding sine waves with frequencies f_1 & f_2 gives

$$\sin(f_1 2\pi t) + \sin(f_2 2\pi t)$$

$$= 2\sin\left(\frac{f_1+f_2}{2} \cdot 2\pi t\right) \cos\left(\frac{f_1-f_2}{2} \cdot 2\pi t\right)$$

If $|f_1 - f_2|$ is "small" this sounds like a note with pitch $(f_1+f_2)/2$ turning on and off with frequency $|f_1 - f_2|$.

Q: Why not frequency $|f_1 - f_2|/2$?

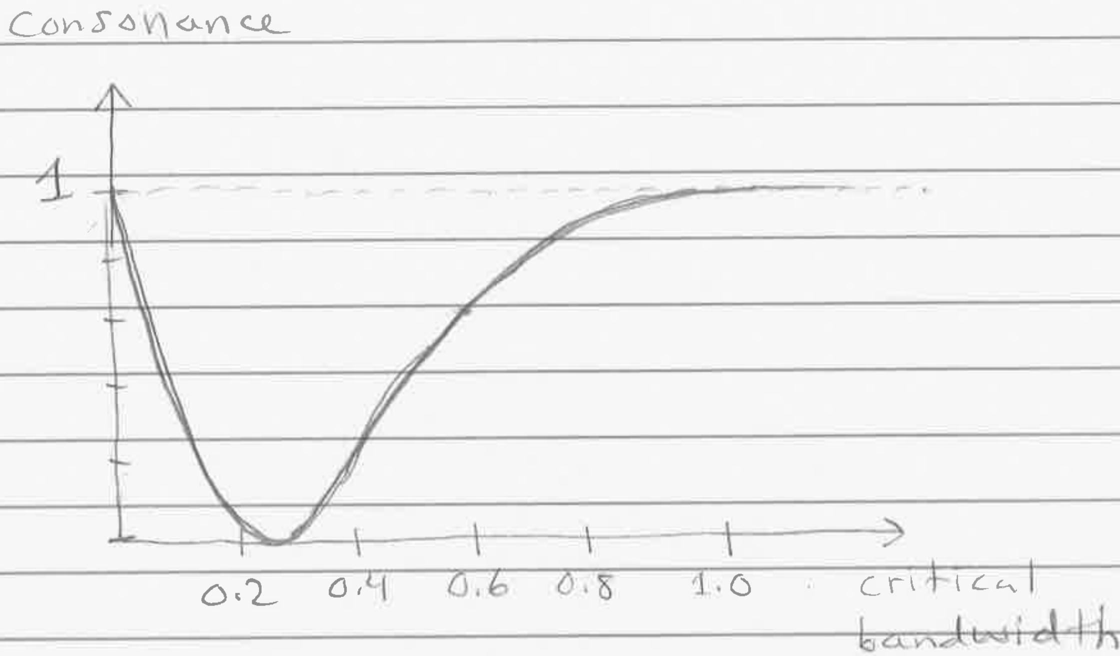
A: Because we are really hearing the absolute value $\left| \cos\left(\frac{f_1 - f_2}{2} \cdot 2\pi t\right) \right|$, which has frequency $|f_1 - f_2|$.

Q: How small is "small"?

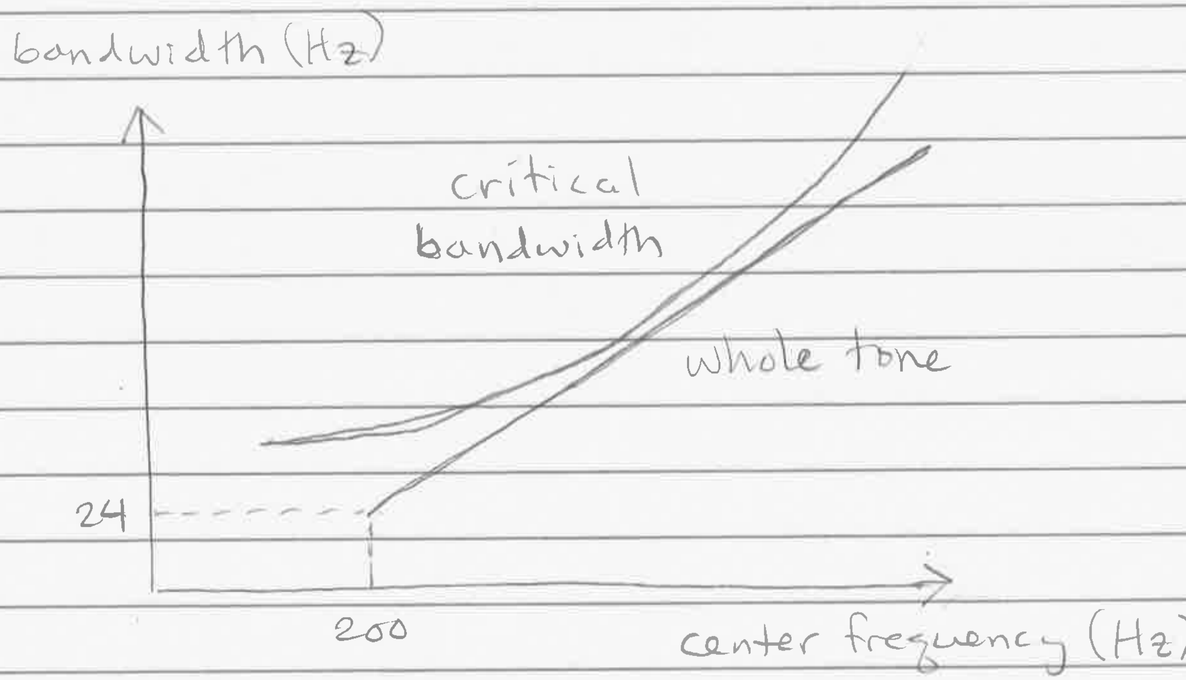
A:
- For reference, the lowest note on the piano has fundamental frequency 27.5 Hz. Anything below that is probably noise.

- von Helmholtz (1863) is known for the theory that dissonance is caused by beats between partials. He claimed that "roughness" is maximized when the frequency difference is around 30-40 Hz, independent of absolute frequency.

- Plomp & Levelt (1965) showed by experiment that roughness is maximized around 1/4 of the "critical bandwidth"



Critical bandwidth arises from the physiology of the ear, and is a function of frequency. It is slightly wider than a whole tone $= 2^{1/6} \approx \underline{1.12}$.



Next time we will use Plomp & Levelt's data to construct a "dissonance curve" showing the relative dissonance of all frequency ratios between 1 and 2 (played on a harmonic instrument such as a string).

They have one in their 1965 paper, but I don't know how they computed it (seeing as it was 1965).

2/5/15

HW 1 due Tues Feb 10
Math Club Mon Feb 9 (Origami)
Quiz 2 Thurs Feb 12

Last time we discussed the mathematics of beats. The superposition of two sine waves of frequencies f_1 & f_2 is

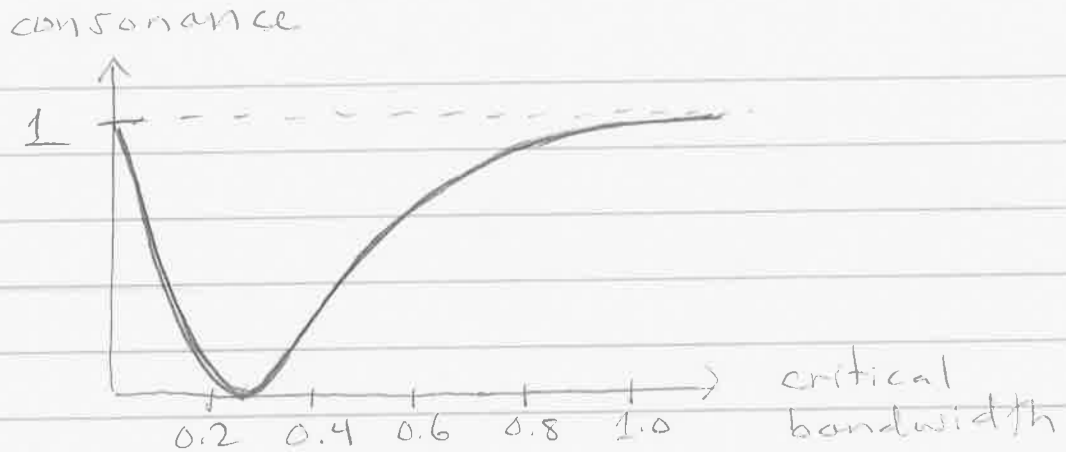
$$\sin(f_1 \cdot 2\pi t) + \sin(f_2 \cdot 2\pi t)$$

$$= 2 \sin\left(\frac{f_1+f_2}{2} \cdot 2\pi t\right) \cos\left(\frac{f_1-f_2}{2} \cdot 2\pi t\right).$$

If $|f_1 - f_2|$ is "small" this sounds like a tone of frequency $(f_1+f_2)/2$ turning on and off $|f_1-f_2|$ times per second.

Q: How small is "small"?

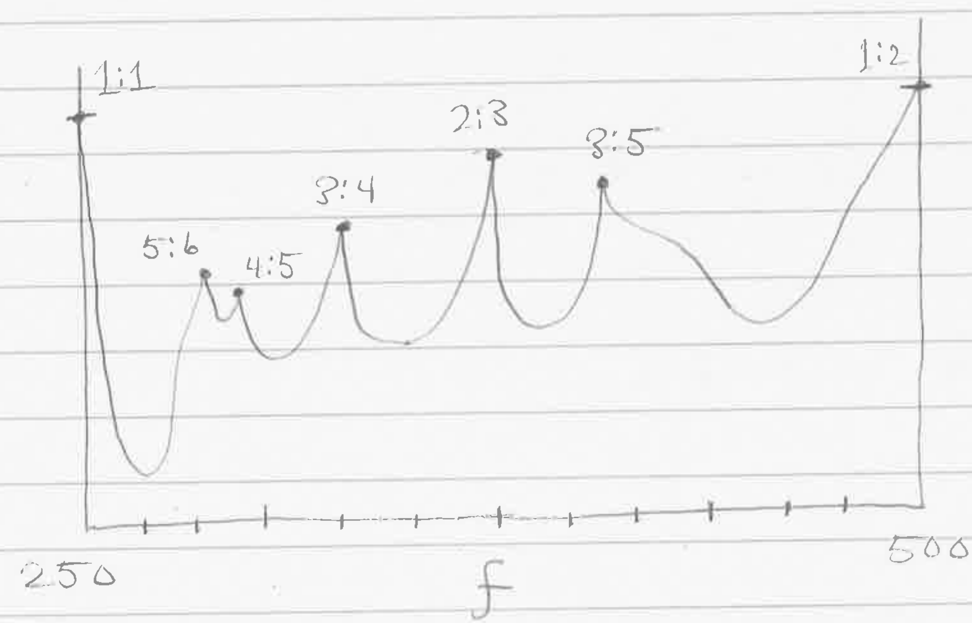
A: Plomp & Levelt (1965) did experiments and determined that "roughness" is maximized when $|f_1-f_2|$ is about 1/4 of the "critical bandwith".



"Plomp-Levelt Curve".

They used this curve to estimate the roughness/dissonance of two harmonic tones played together, based on the roughness between the first 6 partials.

If the two tones have fundamentals 250 Hz and f, the consonance looks like this



They claim that this is experimental verification of Helmholtz' theory of dissonance.

I'm skeptical of how this graph was created so I decided to try it myself.

Assumptions:

- David Benson (pg. 153) suggests formula

$$r(x) = 4|x|\, e^{1-4|x|}$$

for the Plomp-Levelt (roughness) curve, where x is frequency difference in multiples of critical bandwith.

- I'll approximate critical bandwith by a whole tone $= 2^{1/6} \approx 1.12$.

To compute the number of whole tones between frequencies a and b:

$$\frac{\max(a,b)}{\min(a,b)} = \left(2^{1/6}\right)^{?}$$

$$\log\left(\frac{\max(a,b)}{\min(a,b)}\right) = ?\,\log\left(2^{1/6}\right).$$

$$= ?\;\frac{1}{6}\log(2).$$

$$\Longrightarrow\; ? = 6\log\left(\frac{\max(a,b)}{\min(a,b)}\right)\Big/\log(2).$$

I'll call this nwt(a, b) ("number of whole tones").

- For sounds played together we will <u>add</u> the rougness from each pair of partials.

- We will also <u>weight</u> the roughness coming from a pair of partials by their average amplitude. (Plomp & Levelt did not do this, but it seems reasonable. Maybe they should have.)

Here is Maple code to compute the dissonance of an interval.

$f_0 :=$ choose bottom frequency
nump := how many partials do you want to consider ?

```
diss := proc(f)
    if f=f0 then return 0 fi:
    d:= 0
    for i from 1 to nump do
        for j from 1 to nump do
            d:= d + r(nwt(if0, jf))
        od:
    od:
    return d:
end:
```

include $\left(\frac{1}{i}+\frac{1}{j}\right)/2$ if you want to weight the particles.

Then type

```
plot(diss(f), f=f0..2f0);
```

and hope for the best.

[It works! I am impressed that Maple can do this, and I am impressed that Plomp & Levelt did this in 1965.]

To add gridlines corresponding to equal temperament type

```
plot(diss(f), f:f0..2f0,
  axis[1]=[gridlines=[seq(f0·2^(k/12), k=0..12)]]);
```

Compare the equal-tempered scale with what it <u>should</u> be.

Activity:

Look at the handout which shows the dissonance curve based on 8 partials.

Try to label the sharp troughs with small whole number ratios

What do you think of the equal tempered scale?