1/13/15

Welcome to MTH 592!

"The Mathematics of Music"

What *is* this class?

- special topics course
- requested by *you*.
- experimental
- hopefully more courses like this will follow in future semesters.

There is no textbook, I will scan and post all lecture notes on my webpage:

www.math.miami.edu/~armstrong.

My main source is "Music: A Mathematical Offering" by David Benson, available *free* on his webpage.

Prerequisites are *minimal* (just a bit of "mathematical maturity"). I will attempt to entertain a wide variety of students.

Grade will be based on irregular HW assignments and in-class quizzes/exams. There will be NO final exam.
(Since this course is not a pre-requisite for anything, I have no official standard to uphold.)

Course Topic:

A classical education consists of 7 liberal arts, divided into the trivium and quadrivium:

Trivium { grammar, logic, rhetoric }

Quadrivium { arithmetic (number), geometry (number in space), music (number in time), astronomy (number in space & time) }

Since Pythagoras (via Plato), there was recognized a deep analogy between

music & astronomy.

This analogy is expressed via mathematics, and comes in two basic flavors:

(1) Number Theory (Numerology?)
- ratios of small whole numbers
- continued fraction expansions
- this perspective is very old. Some is still relevant; some is obsolete

(2) Trigonometry / Physics
- the mathematics of oscillation
- Fourier analysis
- this is much newer and still very relevant.

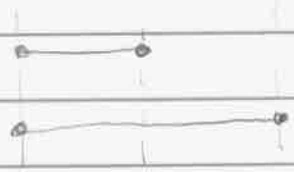
The peculiar character of music theory comes from the interaction of (1) & (2).

Today I will try to sketch the most important interaction, that will motivate everything else.

(1) Number Theory.

The Pythagoreans are credited with the discovery that two strings (of the same composition, under the same tension) will sound good together if their lengths are in the ratio of

small whole numbers.



"octave" very nice

"perfect fifth" rather nice

"perfect fourth" quite nice.

Later we will investigate whether this fact can be explained in a modern way. (Spoiler: It can!)

The desire for small whole numbers puts constraints on the development of musical practice / theory.

<u>Example</u>: The octave (2/1) is the most pleasant (consonant) interval. Psychologically, we perceive the two notes as "the same color". In this sense, the musical spectrum is <u>periodic</u>.

[Compare to the visual spectrum.]

Most cultures in the world divide the octave into 12 intervals. <u>Why</u> do they do this?

The most consonant interval after 2/1 is the perfect fifth (3/2). To create organized music we must know the relationship between octaves and fifths.

In other words we want to solve

$$m \text{ Fifths} = n \text{ octaves}$$

for whole numbers m and n.

$$\left(\frac{3}{2}\right)^m = \left(\frac{2}{1}\right)^n$$

$$3^m = 2^{m+n}.$$

Can this be solved? NO! because 3^m is odd and 2^{m+n} is even.

How about an approximate solution?
Take logarithms

$$\log(3^m) = \log(2^{m+n})$$

$$m \log(3) = (m+n)\log(2)$$

$$\frac{\log(2)}{\log(3)} = \frac{m}{m+n}.$$

In other words we want to approximate the irrational number $\log(2)/\log(3)$ (≈ 0.631) by a fraction.

There is a beautiful way to do this using the continued fraction expansion. It results in a sequence of successively better approximations:

$$\frac{1}{1}, \frac{1}{2}, \frac{5}{8}, \boxed{\frac{12}{19}}, \frac{41}{65}, \frac{53}{84}, \cdots \longrightarrow \frac{\log(2)}{\log(3)}.$$

The approximation $12/19$ is "good enough".

$$\frac{\log(2)}{\log(3)} \approx \frac{12}{19}$$

$$3^{12} \approx 2^{19}$$

$$\left(\frac{3}{2}\right)^{12} \approx \left(\frac{2}{1}\right)^{7}$$

12 fifths $\approx$ 7 octaves.

And that is why we divide the octave into 12 intervals.

The theory of "harmony" is all about finding impossible solutions to impossible number-theoretic problems.

But *why* are small whole number ratios the correct thing to look at?

Ran out of time.

Next time we'll use trigonometry and physics to explain this.

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The history of music theory is closely tied to the history of astronomy.

"music of the spheres".

The analogy is based on two flavors of mathematics:

1) Number Theory / Numerology

2) Trigonometry / Physics

Last time we discussed how the theory of harmony arises from Pythagoras' observation:

☆ Two strings (of the same composition, under the same tension) will sound good together if their lengths are in the ratio of small whole numbers.

We will call this P.O.

"Pythagoras' Observation"

Examples:

$2/1$ octave
$3/2$ perfect fifth
$4/3$ perfect fourth.

[In ancient times only the numbers 1, 2, 3, 4 we considered "small". Today we also think of 5, 6 as "small", sometimes even 7, 8!]

Assuming P.O., we used Number Theory to explain why most cultures divide the octave into 12 intervals

$$m \text{ fifths} \neq n \text{ octaves}$$

$$3^m \neq 2^{m+n}$$

HOWEVER.

$$3^{12} \approx 2^{19}$$

12 fifths $\approx$ 7 octaves

and that's why we have 12 notes. The mathematics of harmony is all about finding approximate solutions to impossible number theory problems.

Discussion:

- But <u>why</u> is P.O. true?
- <u>Is</u> it even true
- What does "sound good" mean?

To explain this we turn to the second (more modern) flavor of musical mathematics.

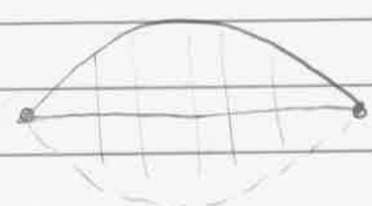
(2) Trigonometry / Physics

We consider a string with fixed ends.

It is observed that the string can only vibrate at certain (resonant) frequencies. All other frequencies tend to cancel themselves out.

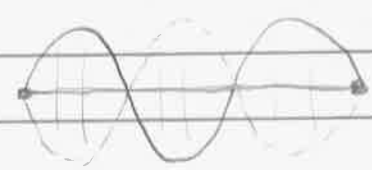
The lowest resonant frequency f is called the fundamental. The other resonant frequencies are then

$$f, 2f, 3f, 4f, \ldots$$



f

$2f$

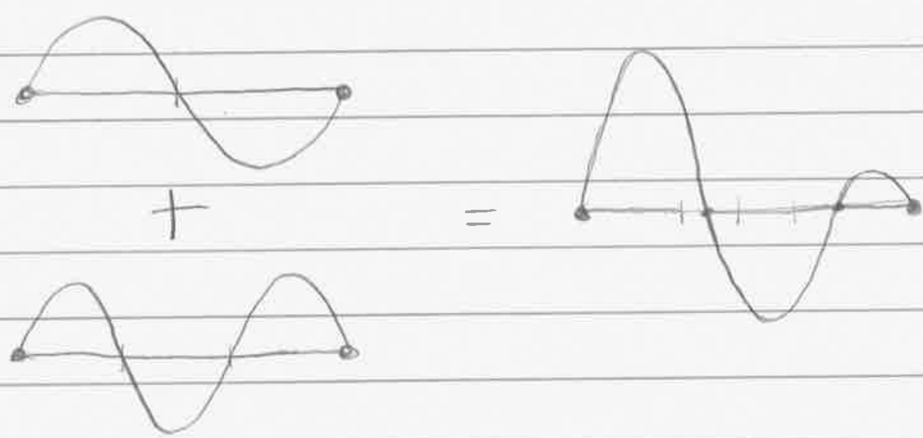


$3f$

"Normal modes" of vibration.

This means that any vibration of the string must be a superposition of normal modes.

[Superposition means we add the displacements at each point.



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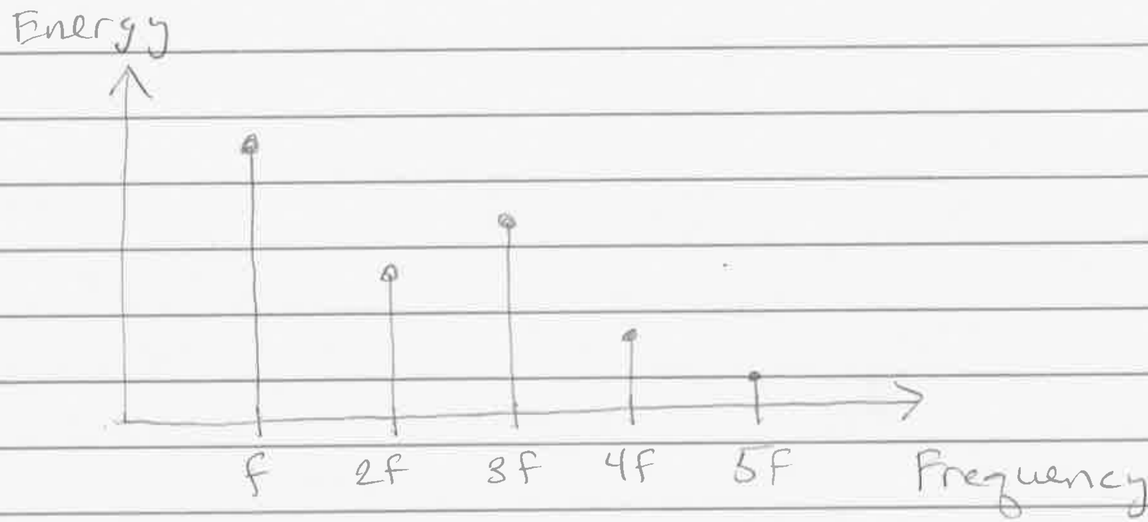
If we record the "amount" of vibration at each resonant frequency (probably as energy) then we obtain a kind of "fingerprint" for the sound.

[The technical name is

"Fourier Transform".]



Since total energy is finite, the energy at high frequencies is very small.

We perceive f as the pitch of the sound. The higher harmonics determine the sound's texture (called its timbre).

Example: The sound of a clarinet is dominated by $f, 3f, 5f, 7f$, etc. Even multiples $2f, 4f, 6f, \ldots$ are canceled by the shape of the instrument (a closed tube).

Back to P.O.

Q: What happens when two strings are played together?

Suppose we have two strings of length l_1 and l_2 with fundamental frequencies f_1 and f_2.

If $\dfrac{l_1}{l_2} = \dfrac{a}{b}$

then how are f_1 and f_2 related?

<u>Some Physics</u> (Dimensional Analysis):

Consider a string with fixed ends

The basic properties of the string are:

l = length
μ = linear density
T = tension

I will use the notation

" property ~ units (in MKS) "

so that

$$\ell \sim \text{meters} = m.$$

$$\mu \sim \text{kilograms}/\text{meter} = kg/m.$$

$$T \sim \;?$$

Tension is a force. Newton's 2nd Law tells us that

$$\text{force} = \text{mass} \times \text{acceleration}$$

$$\sim \text{kilograms} \times \text{meters}/(\text{second})^2$$

$$T \sim kg \cdot m/s^2.$$

we can cancel the kilograms to obtain

$$\frac{T}{\mu} \sim \frac{kg \cdot m/s^2}{kg/m} = \frac{m^2}{s^2}$$

Thus

$$\sqrt{\frac{T}{\mu}} \sim \frac{m}{s} = \text{velocity}.$$

Let's define a new property

$$v := \sqrt{\frac{T}{\mu}}$$

This has the dimensions of velocity.
The velocity of <u>what</u> ?

Vincenzo Galilei (Galileo's father) was a musician in Florence in the 1580's, dedicated to recovering the glory of ancient Greek music. He was the first modern person to subject P.O. to experimental test.

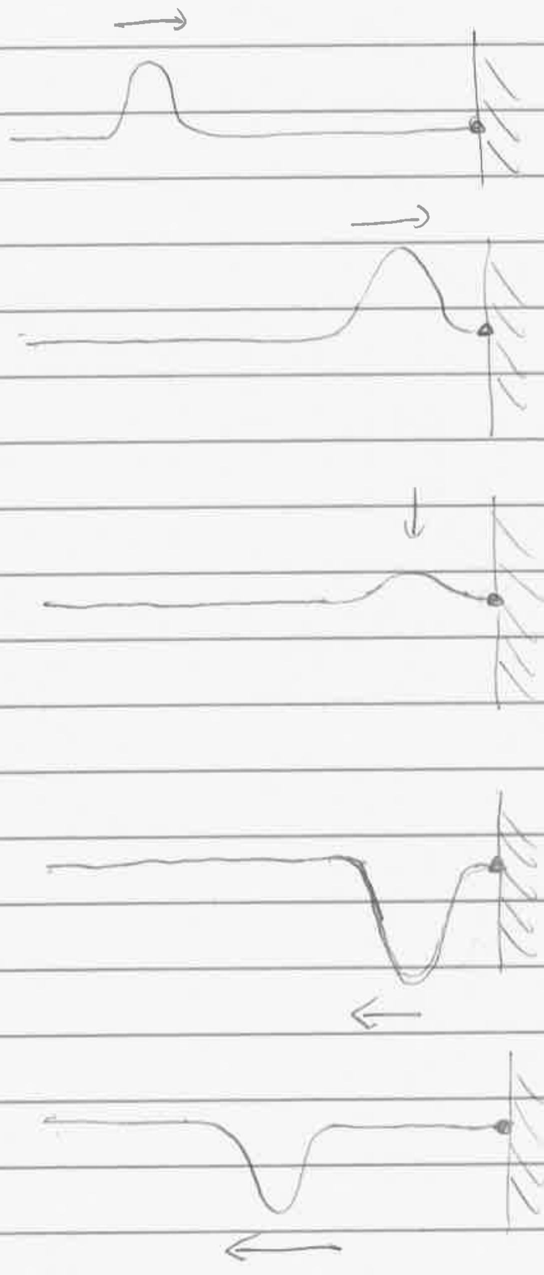
Vincenzo's Observation:

$\sqrt{T/\mu}$ is the velocity of propagation of a disturbance on a string under tension.

Q: Given v how do we compute f ?

[Note: $f \sim 1/\text{second} = 1/s$ (Hz)]

Assuming that a disturbance is "totally reflected" at the fixed ends



How long does it take to repeat itself?

It travels distance $2\ell \sim m$ [why 2ℓ?] at velocity $v \sim m/s$. The time required is

$$\frac{2\ell}{v} \sim \frac{m}{m/s} = s$$

This is called the <u>period</u> of oscillation. The <u>frequency</u> of oscillation is

$$f = \frac{v}{2\ell} \sim \frac{1}{s}$$

We conclude that

$$\boxed{f = \frac{1}{2\ell}\sqrt{\frac{T}{\mu}}}$$

[The fundamental frequency of a string is a function of length, tension and density. The string can be "tuned" by changing any of these quantities.]

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Today I will finish the introduction. After that we'll discuss HW & Quizzes, and begin the course proper.

Recall: We are trying to give a modern justification for Pythagoras' Observation (small whole number ratios "sound good").

Last time we discussed a vibrating string with fixed ends.

If the string has linear density μ (kg/m) and is under tension T ($kg \cdot m/s^2$), then the quantity T/μ is measured in

$$\frac{\cancel{kg} \cdot m/s^2}{\cancel{kg}/m} = \frac{m}{s^2} \cdot \frac{m}{1} = \frac{m^2}{s^2} = \left(\frac{m}{s}\right)^2 .$$

In 1580s Florence, Vincenzo Galilei (Galileo's father) made the following observation.

☆ Vincenzo's Observation:

A disturbance in the string will propagate with velocity $\sqrt{T/\mu}$.

We assume that a disturbance is reflected at the fixed ends with a "phase shift of 180°".

[See animation.]

Suppose the string has length ℓ. Then <u>any</u> disturbance will return to its original position after traveling distance 2ℓ.

At speed $\sqrt{T/\mu}$, this trip will take

$$\frac{2\ell}{\sqrt{T/\mu}} \sim \frac{m}{m/s} = \frac{1}{1/s} = \text{seconds}$$

This is called the <u>period</u> of oscillation.

The frequency of oscillation is

$$f = \frac{\sqrt{T/\mu}}{2l} = \frac{1}{2l}\sqrt{\frac{T}{\mu}},$$

measured in "Hertz" ($1/$seconds).

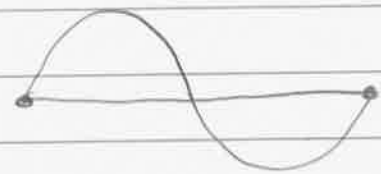
Summary: Any disturbance in the string will oscillate with frequency

$$\boxed{f = \frac{1}{2l}\sqrt{\frac{T}{\mu}}}$$

This is called the fundamental or resonant frequency of the string.

Disturbances with a higher amount of symmetry could possibly oscillate at integer multiples of f.

Q: When will this disturbance return to its starting position?



A: It will return after traleling distance l. Since it travels only half the distance, this will take half the time:

$$\text{period} = \frac{1}{2}\frac{1}{f} = \frac{1}{2f}$$

Thus it will oscillate with frequency

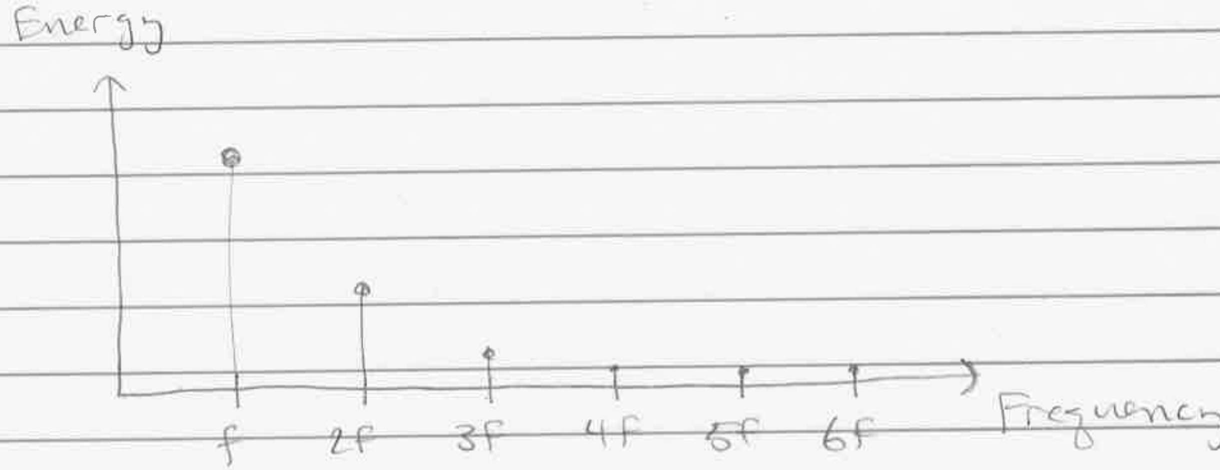
$$2f.$$

☆ The Principle of Superposition:

It is possible that the string can support many disturbances oscillating simultaneously at frequencies

$$f,\ 2f,\ 3f,\ 4f,\ \ldots$$

We get a "fingerprint" of the sound by by recording the amount of energy at each frequency.



"Fourier Transform"

[See examples of sawtooth, triangle wave and square wave on Mathematica.]

Back to P.O.:

What happens if we play two strings at the same time?

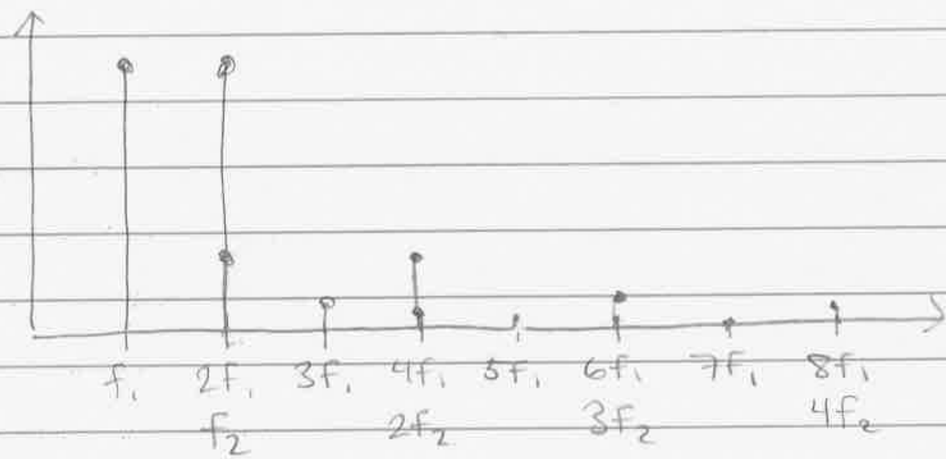
Consider two strings of lengths l_1 & l_2 with fundamental frequencies f_1 & f_2. Assume tension T and density μ are the same for both.

Then we have

$$\frac{f_2}{f_1} = \frac{\sqrt{T/\mu}\ /2l_2}{\sqrt{T/\mu}\ /2l_1} = \frac{1/l_2}{1/l_1} = \frac{l_1}{l_2} .$$

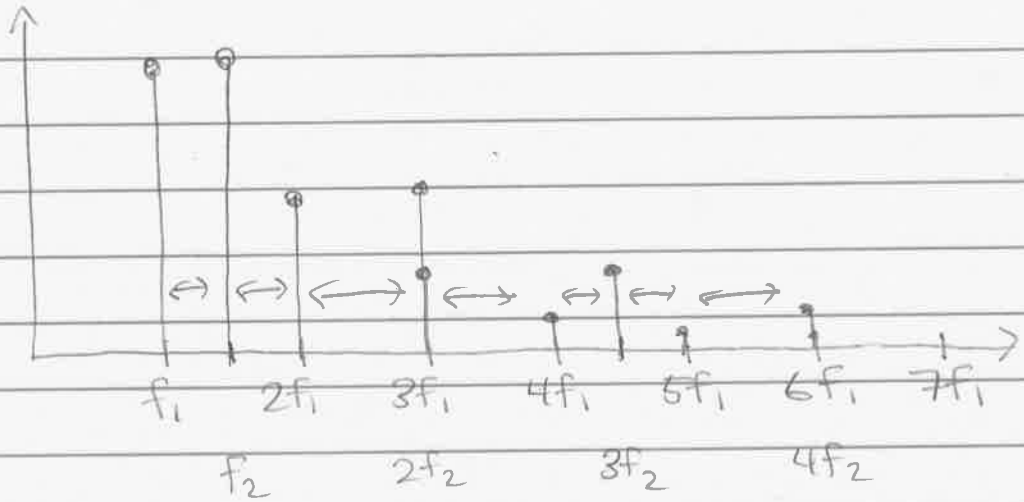
Example: $l_1/l_2 = f_2/f_1 = 2/1$.

The spectra look like.



You can barely hear this as two notes. It just sounds like f_1 with some harmonics increased.

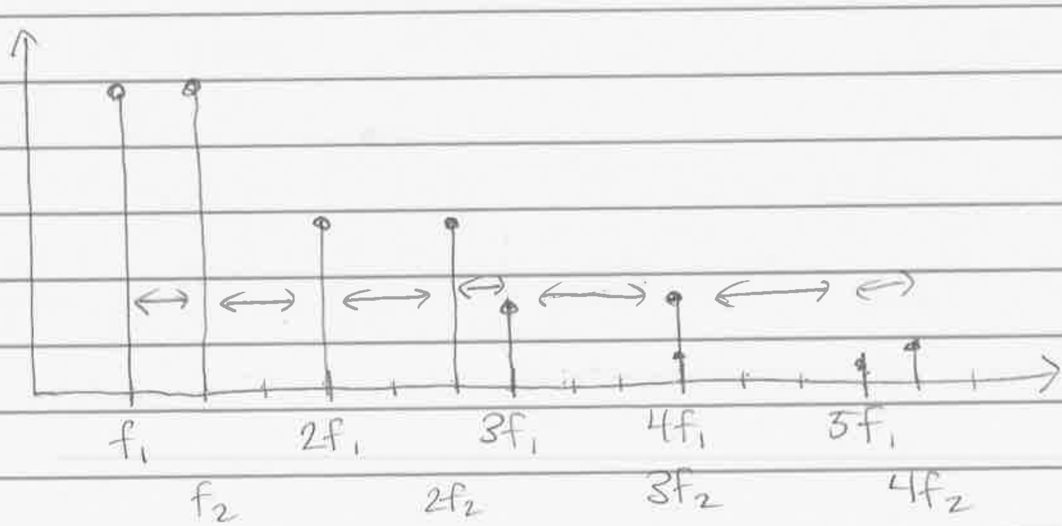
Example: $l_1/l_2 = f_2/f_1 = 3/2$.



What do we see / hear?

- nice orderly pattern.
- partials never get too close together (never closer than $\frac{1}{2} f_1$).

Example: $f_2 / f_1 = 4/3$



- partials never closer than $\frac{1}{3} f_1$.

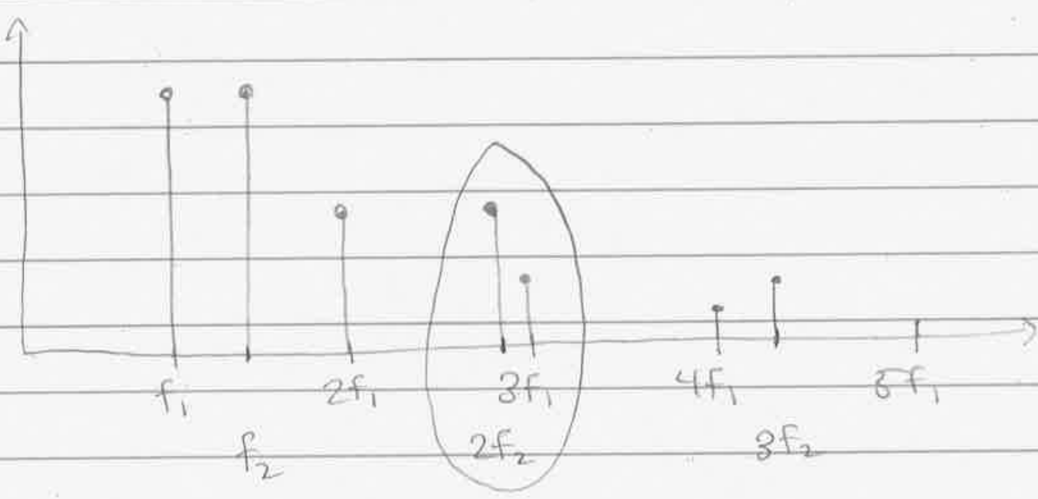
Observation:

If $f_2/f_1 = a/b$, then the partials are never closer than $\frac{1}{b} f_1$.

If b is small, this means the partials are far apart.

Q: Why is it good to keep the partials far apart?

Example: $f_2/f_1 = \sqrt{2} \approx 1.414$



- $3f_1$ and $2f_2$ are rather close. How does it sound?

[Listen to 740 Hz vs. 785 Hz.]