Math 592 **Spring 2015**
Homework 3 **Drew Armstrong**

A Justly Tuned Chromatic Scale.

On this homework you will compute the best rational approximations to the notes of the equal-tempered chromatic scale. The tool you will use is called continued fractions. Any irrational number α can be expressed uniquely in the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.$$

where a_0 is an integer and $a_1, a_2, \ldots$ are positive integers. This is called the continued fraction expansion of α. To save space we will use the notation

$$\alpha = [a_0; a_1, a_2, a_3, \ldots].$$

Given integers p, q we will say that p/q is a best rational approximation of α if the quantity $|p/q - \alpha|$ is minimized among all fractions with denominators less than or equal to q. It is a theorem that the sequence of best rational approximations of α can be read off from the continued fraction expansion, as follows:

- Truncate the continued fraction at the nth place to get $[a_0; a_1, a_2, \ldots, a_n]$,
- Replace a_n by any integer between $a_n/2$ and a_n.

For example, consider the number $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \ldots]$. The sequence of best rational approximations is given by

$$\begin{array}{ccccccccc} [3] & [3;4] & [3;5] & [3;6] & [3;7] & [3;7,8] & [3;7,9] & [3;7,10] & \cdots \\ 3 & \frac{13}{4} & \frac{16}{5} & \frac{19}{6} & \frac{22}{7} & \frac{179}{57} & \frac{201}{64} & \frac{223}{71} & \cdots \end{array}$$

Problem.

(a) Use this method to compute the best rational approximations to the notes of the equal-tempered chromatic scale. That is, for each value $(2^{1/12})^k$ for k from 1 to 12, choose the first fraction in the sequence of best rational approximations that is within 50 cents (half of a semitone) of the correct value. Call it r_k. [Hint: Use WolframAlpha or some other thing to compute the continued fraction expansions.]

(b) Compute the ratios between all pairs of notes separated by seven semitones. That is, compute the ratios

$$\frac{r_8}{r_1}, \frac{r_9}{r_2}, \ldots, \frac{r_{12}}{r_5}, \frac{2r_1}{r_6}, \frac{2r_2}{r_7}, \ldots, \frac{2r_7}{r_{12}}.$$

Which intervals are close to a perfect fifth and which are far away? The bad ones are called **wolf intervals** because they sound like a wolf howling. This is the main weakness of just intonation.

Many tuning systems try to tame the wolf, but it is an impossible problem. There is really no way to have small integer ratios in one key without creating wolf intervals in other keys.

Solutions.

I used Maple to compute the continued fraction expansions. Unfortunately, Maple doesn't know how to compute the best rational approximations so I had to teach it how to truncate the continued fraction expansions properly. It took quite a while and I crashed Maple twice in the process, but eventually I succeeded. Here are the results.

$k = 1$.

$$2^{1/12} = [1; 16, 1, 4, 2, 7, 1, 1, 2, 2, 7, \ldots]$$

Best Rational Approximation	Continued Fraction	Error in Cents
1	[1]	-100.00
10/9	[1;9]	82.40
11/10	[1;10]	65.00
12/11	[1;11]	50.64
13/12	[1;12]	38.57
14/13	[1;13]	28.30
15/14	[1;14]	19.44
16/15	[1;15]	11.73
17/16	[1;16]	4.96
18/17	[1;16,1]	-1.05
53/50	[1;16,1,2]	0.88

$$r_1 = 13/12$$

$k = 2$.

$$2^{2/12} = [1; 8, 6, 31, 1, 2, 2, 2, 10, 3, 1, \ldots]$$

Best Rational Approximation	Continued Fraction	Error in Cents
1	[1]	-200.00
6/5	[1;5]	115.64
7/6	[1;6]	66.87
8/7	[1;7]	31.17
9/8	[1;8]	3.91
28/25	[1;8,3]	-3.80

$$r_2 = 8/7$$

$k = 3$.

$$2^{3/12} = [1; 5, 3, 1, 1, 40, 5, 1, 1, 25, 2, \ldots]$$

Best Rational Approximation	Continued Fraction	Error in Cents
1	[1]	-300.00
4/3	[1;3]	198.05
5/4	[1;4]	86.31
6/5	[1;5]	15.64
13/11	[1;5,2]	-10.79
19/16	[1;5,3]	-2.49

$$r_3 = 6/5$$

$k = 4$.

$$2^{4/12} = [1; 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, \ldots]$$

Best Rational Approximation	Continued Fraction	Error in Cents
1	[1]	-400.00
3/2	[1;2]	301.96
4/3	[1;3]	98.04
5/4	[1;3,1]	-13.68
19/15	[1;3,1,3]	9.24
24/19	[1;3,1,4]	4.44
29/23	[1;3,1,5]	1.30
34/27	[1;3,1,5,1]	-0.91

$$r_4 = 5/4$$

$k = 5$.

$$2^{5/12} = [1; 2, 1, 73, 11, 1, 1, 10, 2, 9, 319, \ldots]$$

Best Rational Approximation	Continued Fraction	Error in Cents
1	[1]	-500.00
3/2	[1;2]	201.96
4/3	[1;2,1]	-1.96
151/113	[1;2,1,37]	1.87

$r_5 = 4/3$ [This one was really dramatic. Why?]

$k = 6$.

$$2^{6/12} = \sqrt{2} = [1; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots]$$

Best Rational Approximation	Error in Cents	
1	[1]	-600.00
3/2	[1;2]	101.96
4/3	[1;2,1]	-101.96
7/5	[1;2,2]	-17.49
17/12	[1;2,2,2]	3.00
24/17	[1;2,2,2,1]	-3.00
41/29	[1;2,2,2,2]	-0.51

$$r_6 = 7/5$$

$k = 7$.

$$2^{7/12} = \sqrt{2} = [1; 2, 147, 5, 1, 3, 5, 4, 4, 1, 1, \ldots]$$

Best Rational Approximation	Continued Fraction	Error in Cents
1	[1]	-700.00
3/2	[1;2]	1.95
223/149	[1;2,74]	-1.92

$r_7 = 3/2$ [This one was really dramatic. Why?]

$k = 8$.

$$2^{8/12} = [1; 1, 1, 2, 2, 1, 3, 2, 3, 1, 3, \ldots]$$

Best Rational Approximation	Continued Fraction	Error in Cents
1	[1]	-800.00
2	[1;1]	400
3/2	[1;1,1]	-98.04
5/3	[1;1,1,1]	84.36
8/5	[1;1,1,2]	13.69
19/12	[1;1,1,2,2]	-4.44
27/17	[1;1,1,2,2,1]	0.91

$$r_8 = 8/5$$

$k = 9$.

$$2^{9/12} = [1; 1, 2, 7, 81, 2, 1, 3, 12, 1, 2, \ldots]$$

Best Rational Approximation	Continued Fraction	Error in Cents
1	[1]	-900.00
2	[1;1]	300
3/2	[1;1,1]	-198.04
5/3	[1;1,2]	-15.64
22/13	[1;1,2,4]	10.79
27/16	[1;1,2,5]	5.87
32/19	[1;1,2,6]	2.49
37/22	[1;1,2,7]	0.03

$$r_9 = 5/3$$

$k = 10$.

$$2^{10/12} = [1; 1, 3, 1, 1, 2, 1, 1, 15, 2, 1, \ldots]$$

Best Rational Approximation	Continued Fraction	Error in Cents
1	[1]	-1000.00
2	[1;1]	200
5/3	[1;1,2]	-115.64
7/4	[1;1,3]	-31.17
9/5	[1;1,3,1]	17.60
16/9	[1;1,3,1,1]	-3.91
25/14	[1;1,3,1,1,1]	3.80
41/23	[1;1,3,1,1,2]	0.79

$$r_{10} = 7/4$$

$k = 11$.

$$2^{11/12} = [1; 1, 7, 1, 9, 1, 15, 5, 1, 14, 2, \ldots]$$

Best Rational Approximation	Continued Fraction	Error in Cents
1	[1]	-1100.00
2	[1;1]	100
9/5	[1;1,4]	-82.40
11/6	[1;1,5]	-50.64
13/7	[1;1,6]	-28.30
15/8	[1;1,7]	-11.73
17/9	[1;1,7,1]	1.05
100/53	[1;1,7,1,5]	-0.88

$$r_{11} = 13/7$$

In summary, here is our justly tuned chromatic scale:

Note	Ratio	Error in Cents
r_0	1/1	0
r_1	13/12	38.57
r_2	8/7	31.17
r_3	6/5	15.64
r_4	5/4	−13.68
r_5	4/3	−1.96
r_6	7/5	−17.49
r_7	3/2	1.95
r_8	8/5	13.69
r_9	5/3	−15.64
r_{10}	7/4	−31.17
r_{11}	13/7	−28.30
r_{12}	2/1	0

We got this by solving a number theory problem. Each interval r_k/r_0 has been optimized to sound as good as possible according to von Helmholtz' theory. However, we took no consideration of the ratios r_i/r_j for other i and j. It could be that some of these intervals sound rather bad.

As a simple test, let's look at all of the intervals separated by seven semitones. We hope that they are close to 3/2. Are they?

Interval	Ratio	Error from 3/2 in Cents
r_8/r_1	96/65	−26.84
r_9/r_2	35/24	−48.77
r_{10}/r_3	35/24	−48.77
r_{11}/r_4	52/35	−16.57
r_{12}/r_5	3/2	0
$2r_1/r_6$	65/42	54.11
$2r_2/r_7$	32/21	27.26
$2r_3/r_8$	3/2	0
$2r_4/r_9$	3/2	0
$2r_5/r_{10}$	32/21	27.26
$2r_6/r_{11}$	98/65	8.86
$2r_7/r_{12}$	3/2	0

Well, that's really terrible. This demonstrates that the fundamental problem of music theory is impossible, i.e., there is no way to tune intervals to small whole number ratios and still be able to play in all keys. Your only options are to mess around with complicated tunings such as quarter-comma meantone, or to go for equal temperament and just put up with the imperfect thirds and sixths.