MTH 562/662 **Spring 2019**
Exam 1 — Wed Feb 20 **Drew Armstrong**

Problem 1. Let R be a ring and let $I \subseteq R$ be an additive subgroup.

(a) State what it means for I to be an *ideal*.

We say that $I \subseteq R$ is an ideal if for all $a \in I$ and $b \in R$ we have $ab \in I$.

(b) If $\varphi : R \to S$ is a ring homomorphism prove that $\ker \varphi \subseteq R$ is an ideal.

Proof. Let $a \in \ker \varphi$ and $b \in R$. Then we have

$$\varphi(ab) = \varphi(a)\varphi(b) = 0\varphi(b) = 0,$$

and hence $ab \in \ker \varphi$ □

(c) If I is an ideal, prove that the following binary operation on R/I is well-defined:

$$(a + I)(b + I) := (ab) + I.$$

Proof. Assume that $a + I = a' + I$ and $b + I = b' + I$. By definition this means that $a - a' \in I$ and $b - b' \in I$. But then since I is an ideal we have

$$ab - a'b' = ab - ab' + ab' - a'b' = a(b - b') + (a - a')b' \in I,$$

and it follows that $ab + I = a'b' + I$. □

Problem 2. Let R be a ring.

(a) Let $I \subseteq R$ be an ideal. Prove that $I = R$ if and only if I contains a unit.

Proof. If $I = R$ then we have $1 \in R$. Conversely, suppose that $u \in I$ for some unit $u \in R$. Since I is an ideal this implies that $1 = uu^{-1} \in I$ and then for all $a \in R$ we have $a = 1a \in I$. It follows that $I = R$. □

(b) Prove that R is a field if and only if it has **exactly two ideals**.

Proof. First assume that R is a field and let I be any ideal. If $I \neq 0R$ then there exists some nonzero element $a \in I$. Since R is a field we know that a is a unit and then we have $I = 1R$. Finally, since $0 \neq 1$ in a field we conclude that $0R \neq 1R$ are the only two ideals of R. Conversely, suppose that $0R \neq 1R$ are the only two ideals of R. Now let $a \in R$ be any nonzero element and consider the nonzero principal ideal $aR \neq 0R$. Since R has exactly two ideals we must have $aR = 1R$ and then it follows from part (a) that a is a unit. Hence R is a field. □

Problem 3. Let $I \subseteq R$ be an ideal and let $I \subseteq A \subseteq R$ be an additive subgroup.

(a) If $A \subseteq R$ is an ideal, prove that $A/I \subseteq R/I$ is an ideal.

Proof. Let $A \subseteq R$ be an ideal and consider elements $a + I \in A/I$ and $b + I \in R/I$. Since $a \in A$ and $b \in R$ and since A is an ideal we have $ab \in A$. It follows that $(a + I)(b + I) = ab + I \in A/I$ as desired. □

(b) If $A/I \subseteq R/I$ is an ideal, prove that $A \subseteq R$ is an ideal.

Proof. Let $A/I \subseteq R/I$ be an ideal and consider elements $a \in A$ and $b \in R$. Since $a+I \in A/I$ and $b+I \in R/I$ and since A/I is an ideal we have $ab+I = (a+I)(b+I) \in A/I$. This means that $ab+I = c+I$ for some $c \in A$. In summary we have $ab-c \in I \subseteq A$ and $c \in A$. Then since A is closed under addition we conclude that $ab = (ab-c)+c \in A$ as desired. □

(c) Prove that R/I is a field if and only if I is a maximal ideal. [Hint: You may assume the Correspondence Theorem for abelian groups.]

Proof. From 3(a), 3(b) and the Correspondence Theorem for groups we have a bijection

$$\{\text{ideals between } I \text{ and } R\} \longleftrightarrow \{\text{ideals of } R/I\}.$$

Then from 2(b) we have

$$\begin{aligned} I \subseteq R \text{ is maximal} &\iff \#\{\text{ideals between } I \text{ and } R\} = 2 \\ &\iff \#\{\text{ideals of } R/I\} = 2 \\ &\iff R/I \text{ is a field.} \end{aligned}$$

□

Problem 4.

(a) If R is an integral domain, prove that we have $\deg(fg) = \deg(f)+\deg(g)$ for all nonzero polynomials $f(x), g(x) \in R[x]$.

Proof. Suppose that $\deg(f) = m$ and $\deg(g) = n$. By definition this means that

$$\begin{aligned} f(x) &= a_m x^m + \text{lower terms}, \\ g(x) &= b_n x^n + \text{lower terms}, \end{aligned}$$

for some $a_m, b_n \in R$ with $a_m \neq 0$ and $b_n \neq 0$. Multiplying these polynomials gives

$$f(x)g(x) = a_m b_n x^{m+n} + \text{lower terms}.$$

Then since R is a domain we have $a_m b_n \neq 0$ and it follows that $\deg(fg) = m+n$. □

[Remark: If we define $\deg(0) = -\infty$ then the same result holds for all polynomials.]

(b) Give an explicit example of a ring R and polynomials $f(x), g(x) \in R[x]$ such that $\deg(fg) < \deg(f) + \deg(g)$.

Consider the polynomials $f(x) = 2x+1$ and $g(x) = 3x+1$ in the ring $\mathbb{Z}/6\mathbb{Z}[x]$, so that $f(x)g(x) = 6x^2+5x+1 = 0x^2+5x+1 = 5x+1$. Note that $\deg(fg) = 1 < 1+1 = \deg(f)+\deg(g)$.

(c) Give an explicit example of a ring R and polynomials $f(x), g(x) \in R[x]$ satisfying $\deg(f) = \deg(g) = 1$ and $f(x)g(x) = 0$.

Consider the polynomials $f(x) = 2x$ and $g(x) = 3x$ in the ring $\mathbb{Z}/6\mathbb{Z}[x]$, so that $f(x)g(x) = 6x^2 = 0x^2 = 0$. If you accept the definition $\deg(0) = -\infty$ then this example also works for part (b).