5/23

Review of 561/562.

Last time:

- Groups and homomorphisms
- Lagrange's Theorem
- Quotient groups
- Isomorphism Theorems
- Cyclic groups

This time: Representation of Groups

Q: Why is a group operation associative?
A: To model composition of functions.

Let X be a set with structure and consider the set of auto<u>morphisms</u>

$$\mathrm{Aut}(X) = \{ \text{invertible maps } X \to X \text{ that preserve structure} \}.$$

Examples:

① $\mathrm{Aut}(\text{set } \{1, 2, \ldots, n\}) = S_n$

② $\mathrm{Aut}(\text{group } \mathbb{Z}/n\mathbb{Z}) \approx (\mathbb{Z}/n\mathbb{Z})^\times$

(2)' $\mathrm{Aut}(\text{group } \mathbb{Z}) \approx \{\pm 1\}$.

(3) $\mathrm{Aut}(\text{vector space } \mathbb{R}^n) = GL_n(\mathbb{R})$.

(4) $\mathrm{Aut}(\text{inner product space } \mathbb{R}^n) = O(n)$

$$= \{ A \in GL_n(\mathbb{R}) : A^t A = I \}$$

Note: (1) & (3) are essentially definitions but (2) & (4) are theorems.

Exercise: If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a linear function preserving the standard inner product, prove that

$$f(x) = Ax$$

for some matrix with $A^t A = I$.

Theorem (Cayley): Every abstract group is (isomorphic to a subgroup of) the automorphism group of some structure X.

Proof: Let G be a group and define a map

$$\varphi : G \longrightarrow \mathrm{Aut}(\text{set } G).$$
$$g \longmapsto \left(\begin{array}{l} \varphi_g : G \to G \\ \quad h \mapsto gh \end{array} \right)$$

This is an injective homomorphism ///

But we might have $G \leq \mathrm{Aut}(X)$ for many different structures X.

Philosophy (Representation Theory):

Given an abstract group G, a nice structure X, and a group hom $\varphi : G \to \mathrm{Aut}(X)$ we get a correspondence

info about G	$\longleftrightarrow$	info about X

Felix Klein, 1872

In the simplest case X is just a set (i.e. with no structure)

Let G = a group, X = a set. Then any group hom $\varphi : G \to \mathrm{Aut}(X)$ $(= S_X)$ is called an <u>action</u> of G on X.

Each $a \in G$ gets sent to a self-bijection (= permutation) $\varphi_a : X \to X$.

Exercise: We could equivalently define a group action as a map

$$\begin{aligned} G \times X &\longrightarrow X \\ (a, x) &\longmapsto a * x \end{aligned}$$

satisfying two axioms

- $\forall x \in X$, $1 * x = x$
- $\forall a, b \in G$, $\forall x \in X$, $(ab) * x = a * (b * x)$.

[Hint: $a * x = \varphi_a(x)$]

Notation: Often we will denote a group action by $G \curvearrowright X$ and simply write

$$a * x = a(x).$$

///

Let $G \curvearrowright X$ be an action. For all $x \in X$ we define the orbit

$$\mathrm{Orb}(x) := \{ a(x) : a \in G \} \subseteq X.$$

and the stabilizer

$$\mathrm{Stab}(x) := \{ a \in G : a(x) = x \} \subseteq G$$

Exercise: Prove that

$$x \sim y \iff \exists\, a \in G,\ a(x) = y$$

is an equivalence relation, hence X is a disjoint union of orbits.

Exercise: Prove that $\mathrm{Stab}(x)$ is a subgroup of G. It's probably not normal but we still have Lagrange's Theorem

$$\underbrace{|G/\mathrm{Stab}(x)|}_{\substack{\text{not a group;} \\ \text{it's a "coset space"}}} = |G| / |\mathrm{Stab}(x)|$$

Fundamental Theorem of Group Action
(A.K.A. Orbit-Stabilizer Theorem) :

Let $G \curvearrowright X$ be an action. For each $x \in X$ there is a bijection

$$G/\mathrm{Stab}(x) \longleftrightarrow \mathrm{Orb}(x)$$

Proof: Define
$$\begin{aligned} G/\mathrm{Stab}(x) &\longrightarrow \mathrm{Orb}(x) \\ a\,\mathrm{Stab}(x) &\longmapsto a(x). \end{aligned}$$

Then

$$\begin{aligned} a\,\mathrm{Stab}(x) = b\,\mathrm{Stab}(x) &\iff a^{-1}b \in \mathrm{Stab}(x) \\ &\iff a^{-1}b(x) = x \\ &\iff b(x) = a(x). \end{aligned}$$

$\Longrightarrow$ proves well defined
$\Longleftarrow$ proves injective. ///

We say the action $G \curvearrowright X$ is free if $\mathrm{Stab}(x) = 1$ for all $x \in X$.

Exercise: Then $|G|$ divides $|X|$.

We say $G \curvearrowright X$ is <u>transitive</u> if for all $x, y \in X$ there exists $a \in G$ with $a(x) = y$.

<u>Exercise</u>: Then $|X|$ divides $|G|$.

If $G \curvearrowright X$ is free and transitive (A.K.A. <u>regular</u>) then for any choice of basepoint $x_0 \in X$ we obtain a bijection

$$G = G/\mathrm{Stab}(x_0) \Leftrightarrow \mathrm{Orb}(x_0) = X$$

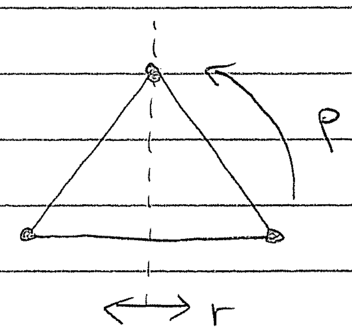
$$G \Leftrightarrow X.$$

<u>Example</u>: Let

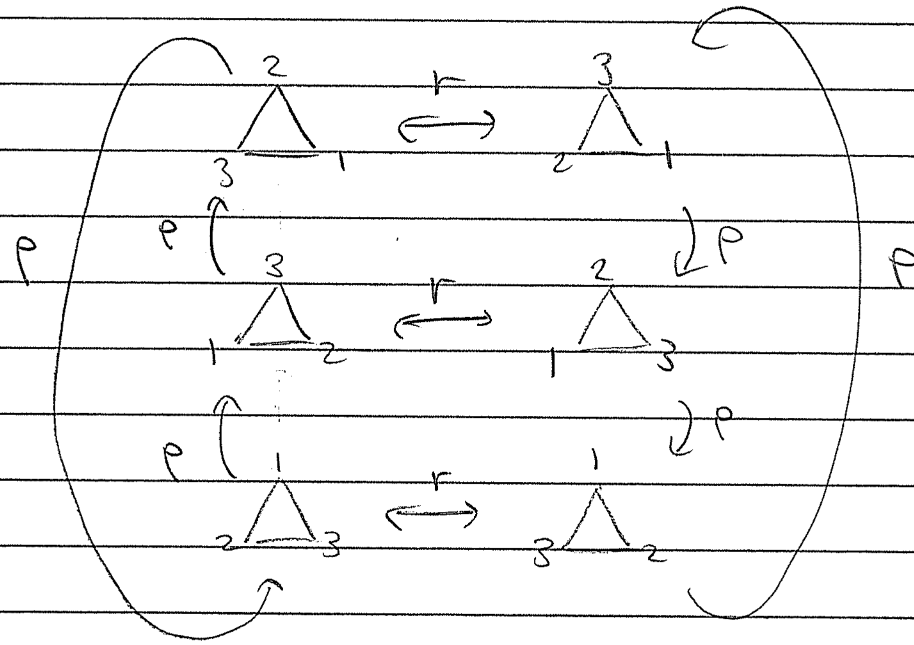
$X = \{$ equilateral triangle with vertices labeled $1, 2, 3$ $\}$

$G =$ The dihedral group

$= \langle \rho, r : \rho^3 = r^2 = 1, r\rho r = \rho^{-1} \rangle$



G acts regularly on X. Picture:



Choose any basepoint to get a bijection $G \leftrightarrow X$. Example:

$$\begin{tikzcd}
r\rho^2 \arrow[r, leftrightarrow, "r"] \arrow[dd, bend right=60, "\rho"'] & \rho^2 \arrow[d, "\rho"] \\
r \arrow[u, "\rho"] \arrow[r, leftrightarrow, "r"] & 1 \arrow[d, "\rho"] \\
r\rho \arrow[u, "\rho"] \arrow[r, leftrightarrow, "r"] & \rho \arrow[uu, bend right=60, "\rho"']
\end{tikzcd}$$