Math 562 **Spring 2014**
Homework 1 **Drew Armstrong**

Problems on Integers

1. The Division Algorithm. Consider integers $a, b \in \mathbb{Z}$ with $b \neq 0$.

(a) Prove that there exist integers $q, r \in \mathbb{Z}$ such that $a = qb + r$ and $0 \leq r < |b|$. [Hint: Let S be the set of integers of the form $a - qb$ for some $q \in \mathbb{Z}$. By well ordering, the set S has a smallest nonnegative element which we can call r. Show that r is small enough.]

(b) Prove that the integers q, r from part (a) are unique. [Hint: Suppose that $a = q_1 b + r_1 = q_2 b + r_2$ with $0 \leq r_1 < |b|$ and $0 \leq r_2 < |b|$. Show that the assumption $r_1 - r_2 \neq 0$ leads to a contradiction.]

(c) Use the Division Algorithm to prove that the equation $2x = 1$ has no solution $x \in \mathbb{Z}$.

2. Application of Unique Factorization.

(a) Consider $a, p \in \mathbb{Z}$ with p prime and $a \neq 0$. Prove that p occurs an even number of times in the prime factorization of a^2.

(b) Use part (a) to give a short proof that $\sqrt{2}$ is irrational. [Hint: Assume for contradiction that there exist $a, b \in \mathbb{Z}$ with $b \neq 0$ and $a/b = \sqrt{2}$.]

Problems on Rings

3. Properties of subtraction.

(a) Given $a \in R$ the axioms say that there exists $a' \in R$ such that $a + a' = 0$. Prove that this a' is unique. We will call it $-a$. Then we define the operation of **subtraction** by

$$a - b := a + (-b).$$

(b) Prove that $a0 = 0$ for all $a \in R$.

(c) Prove that for all $a, b \in R$ we have $(-a)b = -(ab)$. [Hint: Use part (b).]

(d) Prove that for all $a, b \in R$ we have $(-a)(-b) = ab$. [Hint: Use part (c) to show that $ab + a(-b) = 0$. Then use (b).] If a child asks you **why** negative times negative is positive, now you will know what to say.

(e) Prove that for all $a, b, c \in R$ we have $a(b - c) = ab - ac$. [Hint: Use part (c).]

4. Let $\varphi : R \to S$ be a ring homomorphism.

(a) Prove that $\varphi(0_R) = 0_S$.

(b) Prove that $\varphi(-a) = -\varphi(a)$ for all $a \in R$.

(c) Let $a \in R$. If a^{-1} exists, prove that $\varphi(a)$ is invertible with $\varphi(a)^{-1} = \varphi(a^{-1})$.

5. Let R be a ring. We say that $a \in R$ is nilpotent if $a^n = 0$ for some n. If a is nilpotent, prove that $1 + a$ and $1 - a$ are units (i.e. invertible).

6. Let $I \leq R$ be an ideal. Prove that $I = R$ if and only if I contains a unit.

7. Given an ideal $I \leq R$ and an element $a \in R$ we define the additive coset

$$a + I := \{a + x : x \in I\}.$$

Now consider $a, a', b, b' \in R$ such that $a + I = a' + I$ and $b + I = b' + I$. Prove that $(a + b) + I = (a' + b') + I$ and $(ab) + I = (a'b') + I$. This shows that addition and multiplication of cosets is well-defined.