Review of 561/562

(4) Subgroups of GL

DEF: Let $(V, +, \vec{0})$ be an abelian group. The set

$$\operatorname{End}(V) := \{ \text{functions } V \to V \}$$

is a ring (the endomorphism ring) with $+$ and composition.

Let F be a field with a ring hom

$$\varphi : F \longrightarrow \operatorname{End}(V).$$
$$\alpha \longmapsto \varphi_\alpha : V \to V$$

Then (V, F, φ) is a vector space.

Ex. ("the" example). Given field F,

let $V = F^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_i \in F \right\}$

let $\varphi_\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$ "scalar multiplication"

Given vector spaces V, W over F.

We say map $\varphi : V \to W$ is <u>F-linear</u> if

$$\varphi(\alpha \vec{x} + \beta \vec{y}) = \alpha \varphi(\vec{x}) + \beta \varphi(\vec{y})$$

a "vector space map"

Then

$\mathrm{End}(V/F) = \{ F\text{-linear maps } V \to V \}$

$\mathrm{Aut}(V/F) = \{ F\text{-linear bijections } V \to V \}$

<u>Ex.</u> Let $V = F^n$. Then every matrix $A \in \mathrm{Mat}_n(F)$ gives an F-linear map

$$\begin{aligned} \varphi_A : V &\longrightarrow V \\ \vec{x} &\longmapsto A\vec{x} \end{aligned}$$

$$\begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n.$$

In fact, $\varphi : \mathrm{Mat}_n(F) \to \mathrm{End}(F^n)$
$$A \longmapsto \varphi_A$$
is a <u>bijection</u>.

Q: Structure?

☆ Theorem / Definition ☆

$$\varphi_A \circ \varphi_B = \varphi_{AB}$$ "matrix multiplication"

i.e. $\mathrm{End}(F^n) \cong \mathrm{Mat}_n(F)$ as rings

$\mathrm{Aut}(F^n) \cong GL_n(F)$ as groups

$= \{A \in \mathrm{Mat}_n(F) : \det A \neq 0\}$.

"general linear group"

If $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ we can define an <u>inner product</u>

$$\langle \vec{x}, \vec{y} \rangle := \vec{x}^T \bar{\vec{y}} = (x_1 \cdots x_n) \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{pmatrix}$$

$$= x_1 \bar{y}_1 + x_2 \bar{y}_2 + \cdots + x_n \bar{y}_n$$

length $\|\vec{x}\| := \sqrt{\langle \vec{x}, \vec{x} \rangle}$

& distance $d(\vec{x}, \vec{y}) := \|\vec{x} - \vec{y}\|$.

Then we have

$\mathrm{Aut}(F^n, \langle , \rangle) = O(F^n)$

$= \{A \in M_n(F) : A^{-1} = \bar{A}^T\}$

"orthogonal group"

$\leq GL(F^n)$

Notation:

$O(\mathbb{R}^n) = O(n)$ orthogonal
$O(\mathbb{C}^n) = U(n)$ unitary
$O(\mathbb{H}^n) = Sp(n)$ symplectic

☆ NonTrivial Theorem (Cartan-Diudonné) ☆

Every element $A \in O(F^n)$ is a composition of $\leq n$ reflections

Ex. $O(\mathbb{R}^3) = O(3)$

# reflections	det	geometry
0	+1	identity
1	-1	reflection
2	+1	rotation
3	-1	screw reflection

$SO(3)$

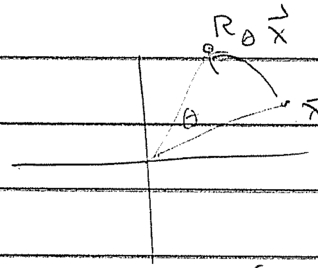
THE END.

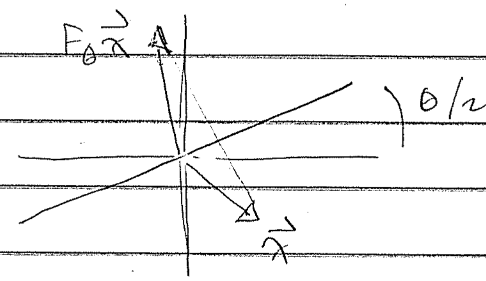
Corollary (Euler's Rotation Theorem):
In $\mathbb{R}^3$ we have

"rotation $\circ$ rotation = rotation"
(or id)

Ex. $O(2) = \{ R_\theta, F_\theta : \theta \in \mathbb{R} \}$

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad F_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$



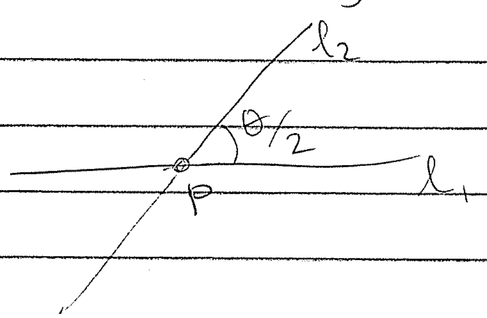


rotate by θ
c. clockwise

reflect across line
of angle $\theta/2$

Note: $F_\alpha F_\beta = R_{\alpha-\beta}$

More Generally



$$F_{l_2} \circ F_{l_1} = R_\theta^p$$

rotate c.c.w.
by θ around p

$$SO(2) = \{ A \in O(2) : \det A = +1 \}$$
$$= \{ R_\theta : \theta \in \mathbb{R} \}$$

Note. $U(1) \approx SO(2)$

$$(e^{i\theta}) \longleftrightarrow R_\theta$$

Appendix: $\mathrm{Isom}(\mathbb{R}^n)$

$\mathrm{Isom}(\mathbb{R}^n)$ = isometries $\mathbb{R}^n \to \mathbb{R}^n$

- 2 special subgroups

① Translations $\mathbb{R}^n_+ := \{t_\alpha : \alpha \in \mathbb{R}\}$
where $t_\alpha(x) := x + \alpha$.
② Linear isometries $\mathrm{Isom}_0(\mathbb{R}^n) = \{f \in \mathrm{Isom}$ with $f(0) = 0\} \approx O(n)$ by Cartan-Dieudonné.

- Given $f \in \mathrm{Isom}$, let $\alpha = f(0) \in \mathbb{R}^n$,
so $\varphi = t_{-\alpha} \circ f \in \mathrm{Isom}_0$
$\Longrightarrow f = t_\alpha \circ \varphi$

- $\mathbb{R}^n_+ \trianglelefteq \mathrm{Isom}$, since $\forall\, t_\alpha \in \mathbb{R}^n_+$ we have

$$
\begin{aligned}
(t_\beta \circ \varphi) \circ t_\alpha \circ (t_\beta \circ \varphi)^{-1}
&= t_\beta \circ \boxed{\varphi \circ t_\alpha \circ \varphi^{-1}} \circ t_{-\beta} \\
&= t_\beta \circ t_{\varphi(\alpha)} \circ t_{-\beta} \\
&= t_{\cancel{\beta + \varphi(\alpha) - \beta}} \qquad = t_{\varphi(\alpha)} \quad \checkmark
\end{aligned}
$$

- $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n_+ \rtimes O(n)$

natural action