Review of 561/562

1. Structure of Groups
2. Symmetry
3. Structure of Rings
4. Other

Today: 2

Q: Why is a "group" operation associative?
A: To model composition of functions!

DEF: Let X = set with structure. Then

$\mathrm{Aut}(X) :=$ group of invertible structure-preserving maps $X \to X$ under composition

$=$ "symmetries" of X.

Examples:

$\mathrm{Aut}(\text{set } \{1, 2, \ldots, n\}) = S_n$

$\mathrm{Aut}(\text{group } \mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$

$\mathrm{Aut}(\text{ring } \mathbb{Z}/n\mathbb{Z}) = \{id\}$

$\mathrm{Aut}(\text{vector space } \mathbb{R}^n) = GL_n(\mathbb{R})$

$\mathrm{Aut}(\text{inner product space } \mathbb{R}^n) = O(n)$.

Philosophy (Representation Theory):

Given abstract group G, study group homs $\varphi : G \to \mathrm{Aut}(X)$ for nice X. Then

info about G	$\longleftrightarrow$	Info about X.

Prototype: Let X = a set (no structure).
G = abstract group.

<u>DEF</u>: A group hom $\varphi : G \to \mathrm{Aut}(X)$ is called an <u>action</u> of G on X (say $G \underset{\varphi}{\curvearrowright} X$).

i.e. send $a \in G$
to a bijection $\varphi_a : X \to X$.

⋮

☆ Exercise: It's equivalent to consider a map

$$\left.\begin{array}{l} G\times X \longrightarrow X \\ (a,x) \longmapsto a*x \end{array}\right\} \text{ satisfying}$$

(A1) $\forall x\in X, \quad e*x = x$

(A2) $\forall a,b\in G, \forall x\in X, \quad (ab)*x = a*(b*x)$.

[Hint: $\varphi_a(x) = a*x$]

Fund. Thm. of Group Action

Given $G \curvearrowright X$ we define

(1) $\forall x\in X, \ \mathrm{Orb}(x) := \{a*x : a\in G\} \subseteq X$

(Exercise: $x\sim y \iff \exists a\in G, \ a*x=y$ is an equivalence. Hence $X = \bigsqcup$ orbits.)

(2) $\forall x\in X, \ \mathrm{Stab}(x) := \{a\in G : a*x = x\} \subseteq G$

(Exercise: $\mathrm{Stab}(x) \leq G$ is a subgroup.)
Probably <u>not</u> normal

But still $\quad \left|G/\mathrm{Stab}(x)\right| = |G|/|\mathrm{Stab}(x)|$

NOT a group

Theorem (Orbit-Stabilizer): $\forall x \in X$

$\exists$ a natural bijection.

$$\begin{array}{ccc} \mathrm{Orb}(x) & \longrightarrow & G/\mathrm{Stab}(x) \\ a*x & \longmapsto & a\,\mathrm{Stab}(x) \end{array}$$

Proof: $a*x = b*x \iff x = (a^{-1}b)*x$

$\iff a^{-1}b \in \mathrm{Stab}(x)$

$\iff a\mathrm{Stab}(x) = b\mathrm{Stab}(x)$

$\Longrightarrow$ well-defined ✓

$\Longleftarrow$ injective ✓

surj. is automatic ✓ $\boxed{\diagup\!\!\diagup}$

Moreover... Given $G \curvearrowright X$. Then $\forall x \in X$

$G \overset{\varphi}{\curvearrowright} \mathrm{Orb}(x)$ by $\varphi_a(b*x) := (ab)*x \quad \forall b \in G$.

$G \overset{\mu}{\curvearrowright} G/\mathrm{Stab}(x)$ by $\mu_a(b\mathrm{Stab}(x)) := (ab)\mathrm{Stab}(x) \quad \forall b$

Exercise: $\forall a \in G, x \in X$ we have

$$\begin{tikzcd} \mathrm{Orb}(x) \arrow[r] \arrow[d, "\varphi_a"'] & G/\mathrm{Stab}(x) \arrow[d, "\mu_a"] \\ \mathrm{Orb}(x) \arrow[r] & G/\mathrm{Stab}(x) \end{tikzcd}$$

$$\boxed{\begin{array}{c} \mathrm{Orb}(x) \approx_G G/\mathrm{Stab}(x) \\ \uparrow \\ \text{as "}G\text{-sets"} \end{array}}$$

Corollary: If $G \curvearrowright X$ transitively (only one orbit) then $\forall x \in X$ we have

$$X \approx_G G/\mathrm{stab}(x)$$

(Philosophy)

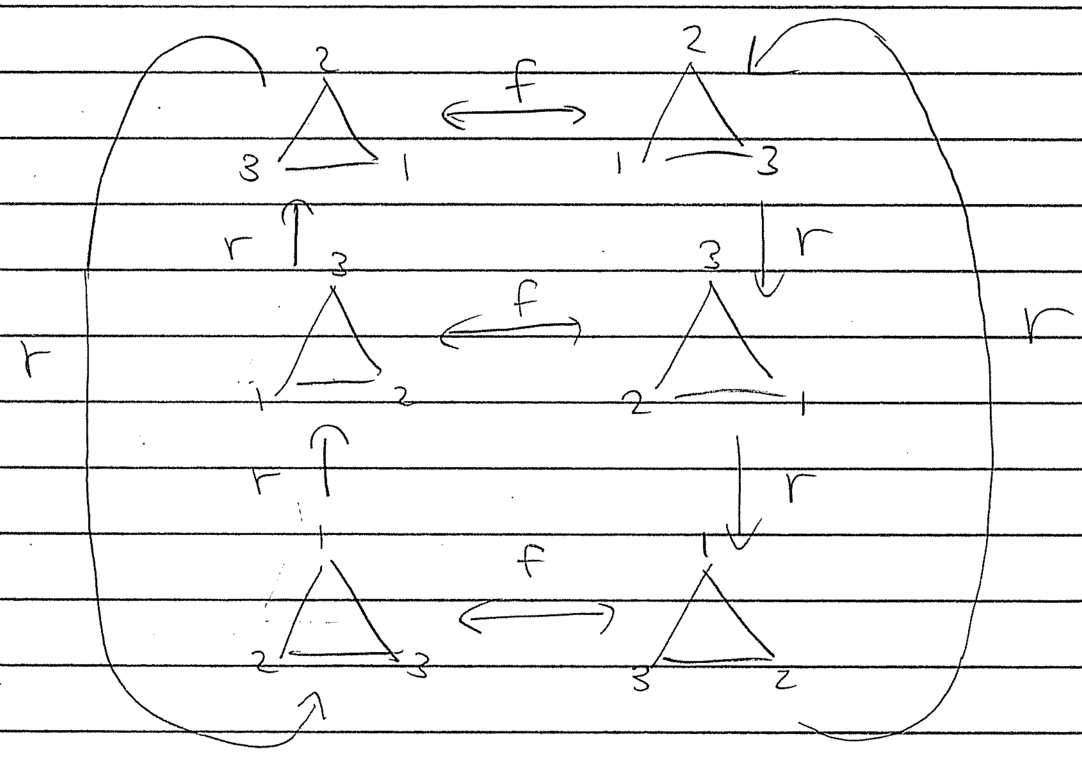
Example:

Let $X = \{\text{labeled triangles}\}$

D_3 = dihedral group

$= \langle r, f : r^3 = f^2 = 1,\ frf = r^{-1} \rangle$

The "Cayley graph"



But $D_3 \curvearrowright X$ with $Stab(x) = \{1\}$ $\forall x \in X$.

So $\boxed{X \cong_{D_3} D_3}$

"Choose a basepoint" to get

$$\begin{tikzcd}
r^2 \arrow[r, leftrightarrow, "f"] \arrow[dd, bend right=60, "r"'] & rf \arrow[d, "r"] \\
r \arrow[u, "r"] \arrow[r, leftrightarrow, "f"] & r^2f \arrow[d, "r"] \\
1 \arrow[u, "r"] \arrow[r, leftrightarrow, "f"] & f \arrow[uu, bend right=60, "r"']
\end{tikzcd}$$

Triangles are unnecessary!

Appendix:

Note. $\langle r \rangle \trianglelefteq D_3$, $\langle f \rangle \leq D_3$
NOT normal

$$\langle r \rangle \cap \langle f \rangle = \{1\}$$

From R.S. (1) we have

$$\langle r \rangle \langle f \rangle := \{ r^i f^j : \forall i,j \in \mathbb{Z} \} = D_3$$

as sets.

Structure? We write

$$D_3 = \langle r \rangle \rtimes \langle f \rangle \approx \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}.$$

In general we have

$$D_n \approx (\mathbb{Z}/n\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

conjugation is inversion

NON-abelian