Today: Review
Friday: Exam 3

First...

Fixed Field Theorem:

Let $K$ be a field with $H \leq \mathrm{Aut}(K)$, $|H| < \infty$.
Then $[K : K^H] = |H|$.

Proof: Given any $\beta \in K$, let $\{\beta = \beta_1, \beta_2, \ldots, \beta_r\}$ be its $H$-orbit. Then

$$g(x) = (x-\beta_1)(x-\beta_2)\cdots(x-\beta_r) \in K^H[x].$$

because $H$ permutes the roots. Then minpoly $\beta / K^H$ has degree dividing $\deg(g) = |\mathrm{orb}_H(\beta)|$, which divides $|H|$ by orbit-stabilizer theorem

$$\left( |\mathrm{orb}_H(\beta)| = |H| / |\mathrm{stab}_H(\beta)| \right)$$

Since $K^H \subseteq K$ is finite and every elt. has degree $\leq |H|$, we find $[K : K^H]$ is finite.

Hence Steinitz says $\exists\, \gamma \in K$ with $K = K^H(\gamma)$.

Let $\mathrm{orb}_H(\gamma) = \{\gamma = \gamma_1, \gamma_2, \dots, \gamma_m\}$. What is $\mathrm{stab}_H(\gamma)$?
If $\mu \in \mathrm{stab}_H(\gamma) \leq H$ i.e. $\mu(\gamma) = \gamma$ and $\mu(a) = a \ \forall\, a \in K^H$, then $\mu(\alpha) = \alpha \ \forall\, \alpha \in K$, i.e. $\mu = id$. Hence $\mathrm{stab}_H(\gamma) = \{id\} \leq H$.

$\Longrightarrow \ m = |\mathrm{orb}_H(\gamma)| = |H| / \underbrace{|\mathrm{stab}_H(\gamma)|}_{1} = |H|$.

It's in $K^H[x]$ for the same reason as $g(x)$

Finally, $h(x) = (x-\gamma_1)(x-\gamma_2)\cdots(x-\gamma_m)$ is the minpoly of $\gamma / K^H$:

- certainly $h(\gamma) = 0$.
- sp. $\exists\, f(x) \in K^H[x]$ with $f(\gamma) = 0$. Then $\forall\, \mu \in H$, $0 = \mu(0) = \mu(f(\alpha)) = f(\mu(\alpha)) = 0$. Hence $f(\gamma_i) = 0 \ \forall\, i \Longrightarrow h(x) \mid f(x)$. ///.

We conclude

$$[K : K^H] = \deg(h(x)) = |H|$$

$\square$

Collecting the results of the semester gives ...

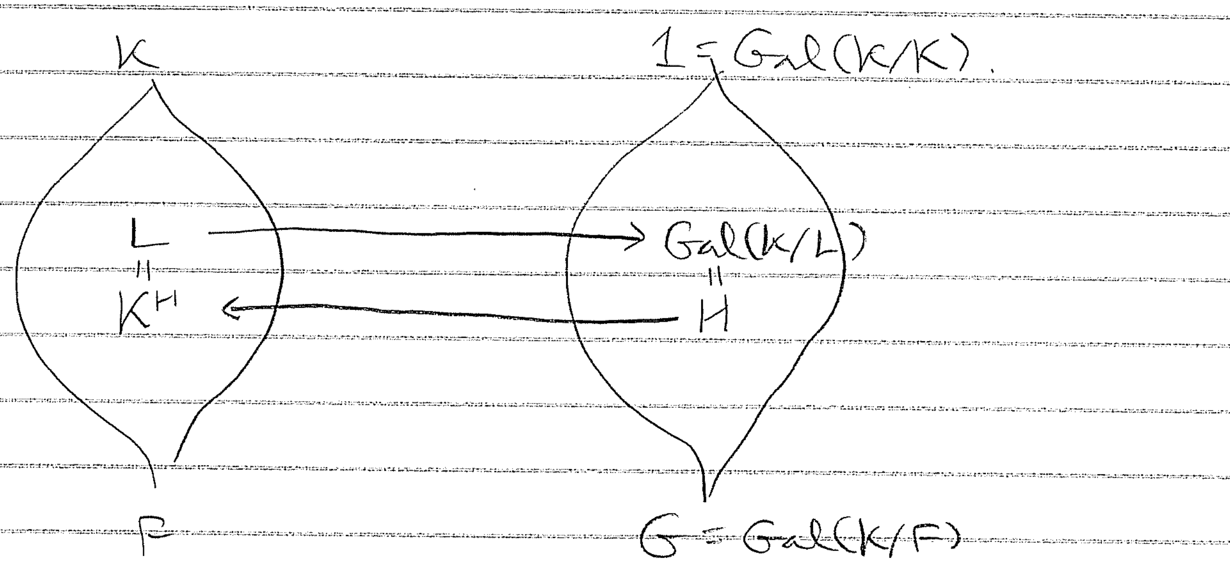
Theorem: Let $K/F$ be finite with $G = Gal(K/F)$. T.F.A.E.

(1) $|G| = [K:F]$
(2) $K^G = F$
(3) $K$ is a splitting field over $F$. ///

If (1), (2), (3) are true we say $K/F$ is "normal" (or "Galois")

Finally, T.F.T.O.G.T.:

Let $K/F$ be normal with $G = Gal(K/F)$. Consider $\mathcal{L}(K/F) = \{L : F \subseteq L \subseteq K\}$ and $\mathcal{L}(G) = \{H : H \leq G\}$. Then the maps $H \mapsto K^H$ and $L \mapsto Gal(K/L)$ are inverse anti-isomorphisms $\mathcal{L}(K/L) \overset{\sim}{\rightleftarrows} \mathcal{L}(G)$



Furthermore we have

$$[K:L] = |H| \quad \text{and} \quad [L:F] = [G:H].$$

Also, $H \trianglelefteq G \iff L/F$ is normal, in which case

$$\mathrm{Gal}(L/F) \cong \frac{\mathrm{Gal}(K/F)}{\mathrm{Gal}(K/L)} \quad \left( = \frac{G}{H} \right)$$

///

Example: The splitting field of $x^4 - 2 \in \mathbb{Q}[x]$ is $\mathbb{Q}(\xi, i)$ where $\xi = \sqrt[4]{2} \in \mathbb{R}_{>0}$

$m_{\xi,\mathbb{Q}}(x) = x^4 - 2$ has roots $\xi, i\xi, -\xi, -i\xi$

$m_{i,\mathbb{Q}}(x) = x^2 + 1$ has roots $i, -i$

$$\Longrightarrow \left|\mathrm{Gal}(\mathbb{Q}(\xi, i)/\mathbb{Q})\right| = 4 \cdot 2 = 8$$

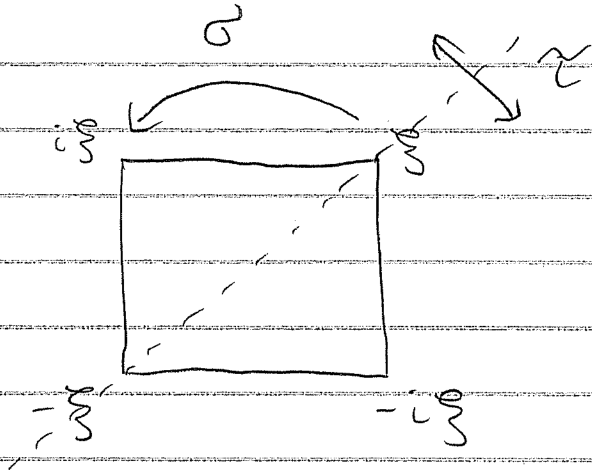
What is it?

Let

| | $\sigma$ | $\tau$ |
|---|---|---|
| $\xi$ | $i\xi$ | $\xi$ |
| $i$ | $i$ | $-i$ |

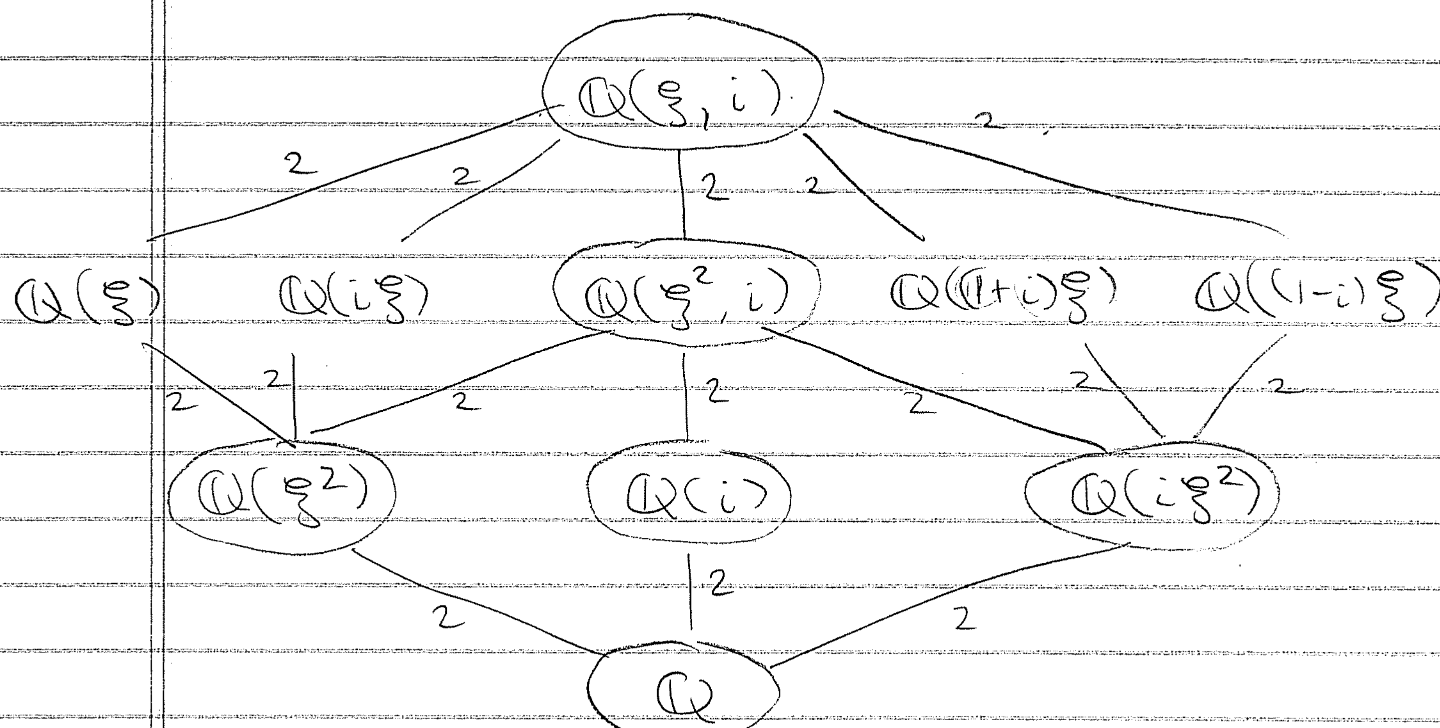
Check: $\sigma^4 = 1$, $\tau^2 = 1$, $\tau\sigma = \sigma^3\tau$

Picture:




$\mathrm{Gal}(\mathbb{Q}(\xi, i)/\mathbb{Q}) \cong D_4$ dihedral

Lattice of Subfields




(L) means $L/\mathbb{Q}$ is normal.