Math 562 **Spring 2012**
Homework 6 **Drew Armstrong**

Problems on Galois Connections

Let S, T be sets and let $R \subseteq S \times T$ be a relation (we will write aRb to denote the statement $(a, b) \in R$). For all subsets $A \subseteq S$ and $B \subseteq T$ let us write

$$A^* := \{t \in T : aRt \ \ \forall a \in A\} \subseteq T,$$
$$B^* := \{s \in S : sRb \ \ \forall b \in B\} \subseteq S,$$

thus defining two functions $* : \wp(S) \to \wp(T)$ and $* : \wp(T) \to \wp(S)$. (Here $\wp(U)$ is the set of subsets of U. The $\wp$ is for "power" set.) We use the symbol $*$ twice for aesthetic reasons; hopefully no confusion will result. Such a pair of maps is called a Galois connection.

1. Prove that for all $A, A' \subseteq S$ and $B, B' \subseteq T$ we have

$$A \subseteq A' \Rightarrow A^* \supseteq A'^* \quad \text{and} \quad B \subseteq B' \Rightarrow B^* \supseteq B'^*.$$

2. Prove that for all $A \subseteq S$ and $B \subseteq T$ we have

$$A \subseteq A^{**} \quad \text{and} \quad B \subseteq B^{**}.$$

3. Prove that for all $A \subseteq S$ and $B \subseteq T$ we have

$$A^{***} = A^* \quad \text{and} \quad B^{***} = B^*.$$

Let X be a set. A function $\mathsf{cl} : \wp(U) \to \wp(U)$ is called a closure operator if it satisfies

- $X \subseteq \mathsf{cl}(X)$ for all $X \subseteq U$,
- $X \subseteq Y \Rightarrow \mathsf{cl}(X) \subseteq \mathsf{cl}(Y)$ for all $X, Y \subseteq U$,
- $\mathsf{cl}(\mathsf{cl}(X)) = \mathsf{cl}(X)$ for all $X \subseteq U$.

(I'm sure you've met at least two examples before.)

4. Prove that $** : \wp(S) \to \wp(S)$ and $** : \wp(T) \to \wp(T)$ are closure operators.

5. Prove that $* : \wp(S) \to \wp(T)$ and $* : \wp(T) \to \wp(S)$ are inverse (and order-reversing) bijections between the $**$**-closed** subsets of S and the $**$**-closed** subsets of T. (They also preserve the "lattice structure" on closed sets, but you don't need to prove this.)

6. Now let $S = K$ be a field and let $T = G$ be a finite group of field automorphisms $G \leq \mathsf{Aut}(K)$. For $a \in K$ and $g \in G$, let aRg mean that $g(a) = a$ (we say g "fixes" a).

(a) Let $F = G^*$. Prove that F is a subfield of K.
(b) For each sub**set** $H \subseteq G$ prove that $F \subseteq H^* \subseteq K$ is an intermediate field.
(c) For each sub**set** $L \subseteq K$ prove that $L^* \subseteq G$ is a subgroup.
(d) Prove that every $**$-closed subset $L \subseteq K$ is an intermediate field $F \subseteq L \subset K$ and every $**$-closed subset $H \subseteq G$ is a subgroup $H \leq G$. (The Fundamental Theorem of Galois Theory says that every intermediate field and every subgroup is $**$-closed. This does not follow from the results of this homework; it requires a careful study of polynomials — cf. everything we did this semester.)