Math 562 **Spring 2012**
Homework 2 **Drew Armstrong**

Problems on General Rings

1. We say that a general ring R is (right) Artinian if every descending chain of (right) ideals terminates. That is, given ideals $R \supseteq I_1 \supseteq I_2 \supseteq \cdots$ there exists some $k \geq 1$ such that $I_k = I_{k+1} = \cdots$. **Prove** that $ab = 1$ implies $ba = 1$ in an Artinian ring. [Hint: Consider the descending chain of right ideals $R \supseteq bR \supseteq b^2R \supseteq \cdots$ (I don't use the notation (b^2) since R is not commutative). Show that there exists some $c \in R$ with $b^k = b^{k+1}c$.]

Problems on Integral Domains

2. Let R be an integral domain. We say that $a, b \in R$ are associates if $b = ua$ where u is a unit. **Prove** that a and b are associates if and only if $(a) = (b)$.

3. We say that a is a proper divisor of b if $b = aq$ and neither a nor q is a unit. **Prove** that a is a proper divisor of b if and only if $(b) < (a) < (1)$ — where " $<$ " means strict containment of ideals.

It's good to have alternate definitions for important concepts. The next two problems show that "integral domain" equals "subring of a field".

4. Let R be a subring of a field (i.e. suppose there exists a field $\mathbb{F}$ and an injective homomorphism $\iota : R \to \mathbb{F}$). **Prove** that R is an integral domain.

5. Let R be an integral domain. Put an equivalence relation on the set R^2 of ordered pairs by saying $(a, b) \sim (c, d)$ if and only if $ad = bc$, and let $[(a, b)]$ denote the $\sim$-class of (a, b). Now define the field of fractions of R,

$$\mathrm{Frac}(R) := \{[(a, b)] : b \neq 0\},$$

with product $[(a, b)] \times [(c, d)] := [(ac, bd)]$ and sum $[(a, b)] + [(c, d)] := [(ad + bc, bd)]$.

(a) Show that $\times$ and $+$ are well-defined on equivalence classes. (It follows that $\mathrm{Frac}(R)$ is a field, but you don't need to verify the boring details.)

(b) Show that the map $\iota(a) := [(a, 1)]$ is an injective homomorphism $\iota : R \to \mathrm{Frac}(R)$.

[Hint: A better name for $[(a, b)]$ might be a/b. Look in the book.]

Problems on Subrings of $\mathbb{C}$

6. Show that the subring $\mathbb{Z}[\sqrt{-2}] := \{a + b\sqrt{-2} : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$ with the norm function $N(a + b\sqrt{-2}) := a^2 + 2b^2$ is a Euclidean domain. [Hint: The principal ideal $(0) \neq (\beta) \subseteq \mathbb{Z}[\sqrt{-2}]$ is a grid of rectangles with side lengths $|\beta|$ and $\sqrt{2}\,|\beta|$. Recall the proof for $\mathbb{Z}[\sqrt{-1}]$.]

7. Consider the subring $\mathbb{Z}[\sqrt{-3}] \subseteq \mathbb{C}$ with norm $N(a + b\sqrt{-3}) := a^2 + 3b^2$.

(a) Show that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in \mathbb{Z}[\sqrt{-3}]$.

(b) Show that $u \in \mathbb{Z}[\sqrt{-3}]$ is a unit if and only if $N(u) = 1$.

(c) Show that $N(\alpha) = 4$ implies that $\alpha \in \mathbb{Z}[\sqrt{-3}]$ is irreducible.

(d) Show that $4 \in \mathbb{Z}[\sqrt{-3}]$ can be factored into irreducibles in **two distinct ways**.

[It follows that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD, hence not a PID, hence not Euclidean. (What happens when you try to generalize your proof from Problem 6?) However $\mathbb{Z}[\sqrt{-3}]$ isn't a total disaster because it "almost" has unique factorization. By "almost" I mean that $\mathbb{Z}[(1 + \sqrt{-3})/2]$ — called the ring of "Eisenstein integers" — is a Unique Factorization Domain. We will see, however, that $\mathbb{Z}[\sqrt{-5}]$ **is** a total disaster.]