Math 562 **Spring 2012**
Homework 1 **Drew Armstrong**

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1. Let R be a ring. We say that $a \in R$ is nilpotent if $a^n = 0$ for some n. If a is nilpotent, prove that $1 + a$ and $1 - a$ are units (i.e. invertible).

2. Let $I \subseteq R$ be an ideal. Prove that $I = R$ if and only if I contains a unit.

3. Let $\varphi : R \to S$ be a ring homomorphism.

(a) Prove that $\varphi(0_R) = 0_S$.
(b) Prove that $\varphi(-a) = -\varphi(a)$ for all $a \in R$.
(c) Let $a \in R$. If $a^{-1} \in R$ exists, prove that $\varphi(a)$ is invertible with $\varphi(a)^{-1} = \varphi(a^{-1})$.

4. Let $I \subseteq R$ be an **ideal** and consider $a, b, c, d \in R$ with $a+I = c+I$ and $b+I = d+I$. Prove that $(a+b)+I = (c+d)+I$ and $ab+I = cd+I$. This shows that addition and multiplication of cosets is well-defined.

5. When does $ab = 1$ imply $ba = 1$? Consider $a, b \in R$ where R is a **finite** ring, and suppose that $ab = 1$. Show that $b + (1 - ba)a^i$ is a right inverse of a for all $i \geq 0$. Use this and the finiteness of R to show that $ba = 1$. [Recall: We have also seen that $AB = I$ implies $BA = I$ for square matrices over a field. Now we have two results of this sort...]

6. Recall that a group G is simple if for **any** group homomorphism $\varphi : G \to H$ we have $\ker \varphi = G$ (the whole group) or $\ker \varphi = 1$ (the trivial group). We can define a simple ring similarly in terms of ring homomorphisms. **Prove** that a ring is simple if and only if it is a field. (Hence the term "simple ring" is unnecessary.) [Hint: Look in the book.]

7. Prove Descartes' Factor Theorem. Let $\mathbb{F}$ be a field and consider the ring $\mathbb{F}[x]$ of polynomials. Given $f(x) \in \mathbb{F}[x]$ and $\alpha \in \mathbb{F}$ such that $f(\alpha) = 0$, prove that $f(x) = (x - \alpha)h(x)$ where $h(x) \in \mathbb{R}[x]$ with $\deg(h) = \deg(f) - 1$. [Hint: Observe that $x^n - \alpha^n = (x - \alpha)(x^{n-1} + \alpha x^{n-2} + \cdots + \alpha^{n-2}x + \alpha^{n-1})$ for all $n \geq 0$. Consider the polynomial $f(x) - f(\alpha)$.]

8. Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex fields. Let $\varphi : \mathbb{R}[x] \to \mathbb{C}$ be the map that sends a polynomial $f(x)$ to its evaluation $f(i) \in \mathbb{C}$ at $x = i$.

(a) Prove that φ is a surjective ring homomorphism.
(b) Recall the definition of complex conjugation: $\overline{a + ib} := a - ib$ for $a, b \in \mathbb{R}$. **Prove** that $f(-i) = \overline{f(i)} \in \mathbb{C}$ for all $f(x) \in \mathbb{R}[x]$.
(c) Use Descartes' Factor Theorem to prove that the kernel of φ is the principal ideal generated by $x^2 + 1$:

$$\ker \varphi = (x^2 + 1) := \{(x^2 + 1)g(x) : g(x) \in \mathbb{R}[x]\}.$$

[This exercise shows that we could replace $\mathbb{C}$ by the quotient ring $\mathbb{R}[x]/(x^2 + 1)$. This is the "grown-up" definition of $\mathbb{C}$.]