Math 562 **Spring 2012**
Exam 1 — Fri Feb 17 **Drew Armstrong**

There are 4 problems and 4 pages. This is a closed book test. Any student caught cheating will receive a score of zero.

1. Let R be an integral domain and consider $a, b, c, p \in R$.

(a) [**1 point**] If $a \neq 0$, prove that $ab = ac$ implies $b = c$.

Proof. Note that $ab = ac \Rightarrow ab - ac = 0 \Rightarrow a(b - c) = 0$. Since $a \neq 0$ this implies that $b - c = 0$, or $b = c$. □

(b) [**2 points**] Prove that if $(a) = (b)$, then a and b are associates.

Proof. Suppose that $(a) = (b)$. This implies that a divides b (say $au = b$) and that b divides a (say $a = bv$). Putting these together gives $a = bv = auv$. Then applying Part (a) gives $1 = uv$. Hence u, v are units and a, b are associates. □

(c) [**2 points**] Prove that if p is prime, then p is also irreducible.

Proof. Suppose that p is prime and that $p = ab$. We want to show that a or b is a unit. Since $p|ab$ we we have (without loss of generality) that $p|a$, say $pk = a$. Substituting this into $p = ab$ gives $p = pkb$, and then applying Part (a) gives $1 = kb$. Hence b is a unit. □

2. Let R be an integral domain.

(a) [**2 points**] What does it mean to say that R is Euclidean? (Give the definition.)

We say that an integral domain R is Euclidean if there exists a function N from nonzero elements of R to positive integers such that: for all $a, b \in R$ with $b \neq 0$ there exist $q, r \in R$ with

- $a = qb + r$,
- $N(r) < N(b)$ or $r = 0$.

(b) [**3 points**] If R is Euclidean, prove that R is a Principal Ideal Domain (PID).

Proof. Let R be Euclidean and consider an ideal $I \subseteq R$. The zero ideal is principal, so suppose that $I \neq (0)$. Then by well-ordering there exists nonzero $a \in I$ with $N(a) \leq N(b)$ for all $b \in I$. Since I is an ideal we have $(a) \subseteq I$. Now consider any $b \in I$ and divide by (nonzero) a to get $b = qa + r$ with $N(r) < N(a)$ or $r = 0$. Since $r = b - qa \in I$, the condition $N(r) < N(a)$ would violate the minimality of a, hence $r = 0$. This implies that $b = qa \in (a)$. Since this true for all $b \in I$ we get $I \subseteq (a)$, hence $I = (a)$ is principal. □

3. (a) [**2 points**] Let R be a ring and consider an increasing chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq R$. Prove that the infinite union $J = \cup_{n=1}^{\infty} I_n$ is also an ideal.

Proof. Let $u, v \in J$ and $a \in R$. By definition there exist n, m such that $u \in I_n$ and $v \in I_m$. Since I_n is an ideal we have $au, ua \in I_n \subseteq J$. Now let $N = \max\{n, m\}$ so that $u, v \in I_N$. Finally since I_N is an ideal we have $u + v \in I_N \subseteq J$. □

(b) [**3 points**] We say that a ring R is Noetherian if there does **not** exist an infinite strictly-increasing chain of ideals $I_1 < I_2 < I_3 < \cdots \subseteq R$. Prove that a PID is Noetherian.

Proof. Let R be a PID and suppose that R is **not** Noetherian; i.e. there exists an infinite chain of ideals $I_1 < I_2 < I_3 < \cdots \subseteq R$. By Part (a) the union $J = \cup_{n=1}^{\infty} I_n$ and ideal and since R is a PID this gives $J = (d)$ for some $d \in R$. Since $d \in J$ there exists N such that $d \in I_N$. But then $J = (d) \subseteq I_N < I_{N+1} \subseteq J$ is a contradiction. Hence R is Noetherian. □

Extra Problem. [**3 points**] We say that a ring R is Artinian if there does **not** exist an infinite strictly-decreasing chain of ideals $R = I_0 > I_1 > I_2 > \cdots \supseteq (0)$. **Prove that an Artinian integral domain is a field**. [Hint: Consider $0 \neq a \in R$ and construct a chain of ideals.]

Proof. Suppose that R is an Artinian integral domain and let $0 \neq a \in R$. Consider the descending chain of ideals $(a) \supseteq (a^2) \supseteq (a^3) \supseteq \cdots \supseteq (0)$. Since R is Artinian there exists n such that $(a^n) = (a^{n+1})$ and then since $a^n \in (a^n) = (a^{n+1})$, there exists $b \in R$ such that $a^n = a^{n+1}b$. Since R is a domain we know that $a \neq 0 \Rightarrow a^n \neq 0$, hence we can cancel a^n from $a^n = a^{n+1}b$ to get $1 = ab$. We conclude that every nonzero element of R is invertible. $\square$