Math 561/661 **Fall 2023**
Homework 2 **Drew Armstrong**

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1. One Step Subgroup Test. Let $(G,*,\varepsilon)$ be a group and let $H\subseteq G$ be a subset. Consider the following four properties:

(S1) $\varepsilon\in H$,
(S2) For all $a,b\in H$ we have $a*b\in H$,
(S3) For all $a\in H$ we have $a^{-1}\in H$,
(S4) For all $a,b\in H$ we have $a*b^{-1}\in H$.

Prove that (S4) holds if and only if all three of (S1), (S2), (S3) hold.

2. Homomorphism and Isomorphism. Consider two groups $(G,*,\varepsilon_G)$ and $(H,\bullet,\varepsilon_H)$. A function $\varphi:G\to H$ is called a *homomorphism* if it satisfies the following condition:

$$\varphi(a*b)=\varphi(a)\bullet\varphi(b)\quad \text{for all } a,b\in G.$$

(a) If $\varphi:G\to H$ is a homomorphism, prove that $\varphi(\varepsilon_G)=\varepsilon_H$.
(b) If $\varphi:G\to H$ is a homomorphism, prove that $\varphi(a^{-1})=\varphi(a)^{-1}$ for all $a\in G$.
(c) Let $\varphi:G\to H$ be a homomorphism and suppose the inverse function φ^{-1} exists. Prove that the function $\varphi^{-1}:H\to G$ is also a homomorphism. It follows that invertible homomorphisms are the same as isomorphisms. [Hint: Given $a,b\in H$, apply the function φ to the group element $\varphi^{-1}(a)*\varphi^{-1}(b)\in G$.]

3. Powers of a Cycle. Consider the "standard 12-cycle" in cycle notation:

$$c:=(1,2,3,4,5,6,7,8,9,10,11,12)\in S_{12}.$$

Compute the first twelve powers $c,c^2,c^3,\ldots,c^{12}$ and express each of them in cycle notation. Try to guess what the k-th power of an n-cycle looks like.

4. Order of an Element. Let $(G,*,\varepsilon)$ be a group and let $g\in G$ be any element. Then for all integers $n\in\mathbb{Z}$ we define the exponential notation

$$g^n:=\begin{cases}\overbrace{g*g*\cdots*g}^{n\text{ times}} & \text{if } n>0,\\ \varepsilon & \text{if } n=0,\\ \underbrace{g^{-1}*g^{-1}*\cdots*g^{-1}}_{-n\text{ times}} & \text{if } n<0.\end{cases}$$

One can prove by induction that this notation satisfies $g^m*g^n=g^{m+n}$ for all $m,n\in\mathbb{Z}$. Equivalently, the function from $(\mathbb{Z},+,0)$ to $(G,*,\varepsilon)$ defined by $n\mapsto g^n$ is a group homomorphism.

(a) Let $\langle g\rangle\subseteq G$ be the smallest subgroup of G that contains the element g. Prove that

$$\langle g\rangle=\{g^n:n\in\mathbb{Z}\}.$$

[Hint: Show that the set on the right is a subgroup of G.]
(b) If $\langle g\rangle$ is a finite set, prove that there exists some $n\geq 1$ such that $g^n=\varepsilon$.
(c) Let $\langle g\rangle$ be a finite set and let $m\geq 1$ be the **smallest positive integer** satisfying $g^m=\varepsilon$. In this case, prove that $\langle g\rangle$ has exactly m elements:

$$\langle g\rangle=\{\varepsilon,g,g^2,\ldots,g^{m-1}\}.$$

This m is called the *order* of the element $g \in G$. If the set $\langle g \rangle$ is infinite then we will say that g has *infinite order*. [Hint: For each integer $k \in \mathbb{Z}$ there exist **unique** integers $q, r \in \mathbb{Z}$ satisfyiing $k = qm + r$ and $0 \leq r < m$.]

5. Join of Two Subgroups. Let G be a group and let $H, K \subseteq G$ be subgroups. Recall that the subgroup generated by the union $H \cup K$ is called the *join*:

$$\begin{aligned} H \vee K &:= \langle H \cup K \rangle \\ &:= \text{ the intersection of all subgroups that contain } H \cup K. \end{aligned}$$

(a) If $(G, +, 0)$ is abelian, we define the *sum* of H and K as follows:

$$H + K := \{h + k : h \in H, k \in K\}.$$

Prove that this is a subgroup.

(b) If $(G, +, 0)$ is abelian, use part (a) to prove that $H \vee K = H + K$.

(c) If $(G, *, \varepsilon)$ is non-abelian, show that the following set is **not** necessarily a subgroup, and hence it does not coincide with the join:

$$H * K := \{h * k : h \in H, k \in K\}.$$

[Hint: The smallest non-abelian group is S_3.]

6. Two Groups with Eight Elements. There are two different non-abelian groups with eight elements, called the *dihedral group* D_8 and the *quaternion group* Q_8. We will use multiplicative notation with identity element called "1".

(a) The dihedral group has elements $D_8 = \{1, r, r^2, r^3, f, rf, r^2f, r^3f\}$ subject to the relations

$$r^4 = f^2 = rfrf = 1.$$

Write out the full 8×8 group table.

(b) The quaternion group has elements $Q_8 = \{1, i, j, k, e, ei, ej, ek\}$ subject to the relations

$$i^2 = j^2 = k^2 = ijk = e, \quad e^2 = 1, \quad \text{and} \quad ae = ea \text{ for all } a \in Q_8.$$

Write out the full 8×8 group table. [If you want you can write e as "-1" and write the elements ei, ej, ek as $-i, -j, -k$, respectively.]

(c) Prove that D_8 and Q_8 are **not isomorphic**. [Hint: Isomorphic groups have the same number of elements of each order. Count the elements of order 2.]