MTH 561/661 **Fall 2023**
Exam 1 — Thurs Sept 21 **Drew Armstrong**

Problem 1. Subgroup Axioms. Let $(G, *, \varepsilon)$ be a group and let $H \subseteq G$ be a subset. Consider the following four properties:

(S1) We have $\varepsilon \in H$.
(S2) For all $a \in H$ we have $a^{-1} \in H$.
(S3) For all $a, b \in H$ we have $a * b \in H$.
(S4) For all $a, b \in H$ we have $a * b^{-1} \in H$.

If (S1), (S2), (S3) hold then we say that H is a *subroup* of G.

(a) If (S1), (S2), (S3) hold, show that (S4) also holds.
(b) If (S4) holds, show that (S1), (S2), (S3) also hold. [Hint: Prove them in order.]

(a): Suppose that (S1), (S2), (S3) hold. In order to show that (S4) holds, consider any $a, b \in H$. From (S2) we have $b^{-1} \in H$ and then from (S3) we have $a * b^{-1} \in H$. □

(b): Suppose that (S4) holds. We will prove (S1), (S2), (S3), in this order.

(S1): Pick any element $a \in H$ (we assume that H is not empty). Then from (S4) we have $\varepsilon = a * a^{-1} \in H$. □

(S2): Consider any $a \in H$. We know that $\varepsilon \in H$ from (S1). Hence from (S4) we have $a^{-1} = \varepsilon * a^{-1} \in H$. □

(S3): Consider any $a, b \in H$. From (S2) we know that $b^{-1} \in H$. Then from (S4) we have $a * b = a * (b^{-1})^{-1} \in H$. □

Problem 2. Group Homomorphisms. Let $(G, *, \varepsilon_G)$ and $(H, \bullet, \varepsilon_H)$ be groups and let $\varphi : G \to H$ be a group homomorphism, i.e., a function satisfying

$$\varphi(a * b) = \varphi(a) \bullet \varphi(b) \text{ for all } a, b \in G.$$

(a) Prove that $\varphi(\varepsilon_G) = \varepsilon_H$.
(b) For all $a \in G$, prove that $\varphi(a^{-1}) = \varphi(a)^{-1}$.

(a): For any group element $a \in G$ we have $\varphi(a) = \varphi(a * \varepsilon_G) = \varphi(a) \bullet \varphi(\varepsilon_H)$. Then multiplying both sides of $\varphi(a) = \varphi(a) \bullet \varphi(\varepsilon_H)$ on the left by the group element $\varphi(a)^{-1}$ (which exists because H is a group) gives $\varphi(\varepsilon_G) = \varepsilon_H$. □

(b): For any group element $a \in G$ we have $\varphi(a) \bullet \varphi(a^{-1}) = \varphi(a * a^{-1}) = \varphi(\varepsilon_G) = \varepsilon_H$, where the last step follows from part (a). Then multiplying both sides of $\varphi(a) \bullet \varphi(a^{-1}) = \varepsilon_H$ on the left by $\varphi(a)^{-1}$ gives $\varphi(a^{-1}) = \varphi(a)^{-1}$. □

Problem 3. Kernel and Image. Let $\varphi : (G, *, \varepsilon_G) \to (H, \bullet, \varepsilon_H)$ be a group homomorphism. Define the subsets $K \subseteq G$ and $M \subseteq H$ as follows:

$K =$ the set of all $g \in G$ such that $\varphi(g) = \varepsilon_H$,

$M =$ the set of all $h \in H$ such that there exists $g \in G$ satisfying $\varphi(g) = h$.

Use the results of Problems 1 and 2 to prove the following.

(a) Prove that K is a subgroup of G.
(b) Prove that M is a subgroup of H.

(a): We will use the one step subgroup test from Problem 1. Consider any elements $a, b \in K$. By definition this means that $\varphi(a) = \varepsilon_H$ and $\varphi(b) = \varepsilon_H$. Then from Problem 2 we have

$$\varphi(a * b^{-1}) = \varphi(a) \bullet \varphi(b)^{-1} = \varepsilon_H \bullet \varepsilon_H^{-1} = \varepsilon_H,$$

which implies that $a * b^{-1} \in K$ as desired. $\square$

(b): We will use the one step subgroup test from Problem 1. Consider any elements $a, b \in M$. By definition this means that we have $a = \varphi(c)$ and $b = \varphi(d)$ for some elements $c, d \in G$. Then from Problem 2 we have

$$a \bullet b^{-1} = \varphi(c) \bullet \varphi(d)^{-1} = \varphi(c * d^{-1}),$$

which implies that $a \bullet b^{-1} \in M$ as desired. $\square$