Exam Wed.
Today: Review

Topics: 2 main ideas.

(1) Symmetries of $\mathbb{R}^n$

(2) Abstract Symmetry
(i.e. group actions).

(1) <u>DEF</u>: A "symmetry" of $\mathbb{R}^n$ is an isometry $f: \mathbb{R}^n \to \mathbb{R}^n$

<u>DEF</u>: $\mathrm{Isom}(\mathbb{R}^n) :=$ group of isometries.

Two subgroups:
Given $\alpha \in \mathbb{R}^n$ define $t_\alpha : \mathbb{R}^n \to \mathbb{R}^n$

$t_\alpha(x) = x + \alpha$. "translation by α".

Given $x, y \in \mathbb{R}^n$,

$\|t_\alpha(x) - t_\alpha(y)\| = \|x + \cancel{\alpha} - (y + \cancel{\alpha})\| = \|x - y\|$.

$\Rightarrow t_\alpha \in \mathrm{Isom}(\mathbb{R}^n)$.

Let $\mathbb{R}^n_+ = \{ t_\alpha : \alpha \in \mathbb{R}^n \}$

Claim: $\mathbb{R}^n_+ \leq \mathrm{Isom}(\mathbb{R}^n)$

Proof:

$t_\alpha \circ t_\beta = t_{\alpha+\beta}$ closed.

$t_0 = id.$ identity

$t_\alpha^{-1} = t_{-\alpha}$ inverses $\square$

$\mathbb{R}^n_+ \approx \mathbb{R}^n$

abelian group w/ vector addition.

DEF: $\mathrm{Isom}_0(\mathbb{R}^n)$ = isometries fixing 0 $\leq \mathrm{Isom}(\mathbb{R}^n)$

Theorem (Cartan-Dieudonné):
Every $f \in \mathrm{Isom}_0(\mathbb{R}^n)$ is a product of $\leq n$ reflections of $\mathbb{R}^n$.

reflection = $\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$

in some basis.

Corollary:

$$\mathrm{Isom}_0(\mathbb{R}^n) \approx O(n).$$

$O(n) = \{ A \in \mathrm{Mat}_n(\mathbb{R}) : A^T A = I \}$

<u>Proof</u>: Show $O(n) \subseteq \mathrm{Isom}_0$ (HW 1.8).

Given $x, y \in \mathbb{R}^n$ we have

$$\|Ax - Ay\|^2 = \|A(x-y)\|^2 = (A(x-y))^T (A(x-y))$$

$$= (x-y)^T \cancel{A^T A}\,(x-y)$$

$$= (x-y)^T (x-y) = \|x-y\|^2 \qquad \blacksquare$$

Show $\mathrm{Isom}_0 \subseteq O(n)$.

$$\mathrm{Isom}_0 = \langle \underset{O(n)}{\underset{\cap}{\text{reflections}}} \rangle \subseteq O(n)$$

$\blacksquare$

So we have $\mathbb{R}^n_+ \leq \text{Isom}$, $O(n) \leq \text{Isom}$.

Observe $\mathbb{R}^n_+ \cap O(n) = \{ id = t_0 = I \}$
trivial

$\Rightarrow$ Every $f \in \text{Isom}$ has $\exists !$ unique expression
$f = t_\alpha \circ \varphi$ with $t_\alpha \in \mathbb{R}^n_+$, $\varphi \in O(n)$.

Proof: Existence.

Let $\alpha = f(0)$ then $t_\alpha^{-1} \circ f(0) = t_\alpha^{-1}(f(0))$
$= t_\alpha^{-1}(\alpha) = \alpha - \alpha = 0$.
$\Rightarrow t_\alpha^{-1} \circ f = \varphi \in O(n)$.
$\Rightarrow f = t_\alpha \circ \varphi$ $\boxtimes$

Uniqueness. Given $\alpha, \beta \in \mathbb{R}^n$, $\varphi, \mu \in O(n)$,

Suppose $t_\alpha \circ \varphi = t_\beta \circ \mu$.
$\Rightarrow t_\beta^{-1} \circ t_\alpha = \mu \circ \varphi^{-1}$
$\Rightarrow \underset{\substack{\cap \\ \mathbb{R}^n_+}}{t_{\alpha-\beta}} = \underset{\substack{\cap \\ O(n)}}{\mu\varphi^{-1}} \in \mathbb{R}^n_+ \cap O(n) = \{1\}$

$\Rightarrow t_{\alpha-\beta} = t_0 \Rightarrow \alpha - \beta = 0 \Rightarrow \alpha = \beta$.

AND $\Rightarrow \mu\varphi^{-1} = I \Rightarrow \mu = \varphi$ $\boxtimes$

what kind of product

$\Rightarrow \quad \mathrm{Isom} = \mathbb{R}^n_+ \cdot O(n)$

Claim: $\mathbb{R}^n_+ \trianglelefteq \mathrm{Isom}$

Proof: Given $\alpha \in \mathbb{R}^n$, $\varphi \in O(n)$ we have

$$\varphi \circ t_\alpha(x) = \varphi(t_\alpha(x)) = \varphi(x+\alpha)$$

$$= \varphi(x) + \varphi(\alpha) = t_{\varphi(\alpha)}(\varphi(x)) = t_{\varphi(\alpha)} \circ \varphi\ (x).$$

$$\Rightarrow \quad \varphi \circ t_\alpha = t_{\varphi(\alpha)} \circ \varphi$$

$$\Rightarrow \quad \varphi \circ t_\alpha \circ \varphi^{-1} = t_{\varphi(\alpha)} \in \mathbb{R}^n_+ \qquad ///$$

Then show $\forall\ t \in \mathbb{R}^n_+$, $f \in \mathrm{Isom}$,

$$f \circ t \circ f^{-1} \in \mathbb{R}^n_+ \qquad \boxtimes$$

Cor: $\mathrm{Isom} = \mathbb{R}^n_+ \rtimes O(n) = \mathrm{Aut}(\mathbb{R}^n)$

Euclidean Geometry

(2) Abstract Symmetry

We say G <u>acts</u> <u>on</u> structure X if $\exists$ group hom

$$\varphi : G \longrightarrow \mathrm{Aut}(X).$$

For us: X is a set (no structure); so

$$\mathrm{Aut}(X) = \underbrace{\{\text{bijections} : X \to X\}}_{\text{"permutations" of } X}.$$

Since $\varphi \mapsto \varphi_g : X \to X$ is a hom we have

(a) $\varphi_1 = id \implies \varphi_1(x) = x \quad \forall x \in X$

$$1 * x = x$$

(b) $\varphi_{gh} = \varphi_g \circ \varphi_h$

$$\implies \varphi_{gh}(x) = \varphi_g(\varphi_h(x)) \quad \forall x \in X$$

$$(gh) * x = g * (h * x)$$

Orbit-Stabilizer Theorem: Given $G \curvearrowright X$. $\forall x \in X$ we have

$$\begin{aligned} \mathrm{Orb}(x) &\longleftrightarrow G/\mathrm{Stab}(x) \\ h*x &\longleftrightarrow h\,\mathrm{Stab}(x) \end{aligned}$$

Moreover, $\forall g \in G$ we have

$$\begin{tikzcd} \underset{h*x}{\mathrm{Orb}(x)} \arrow[r, "f"] \arrow[d, "\varphi_g"'] & \underset{h\,\mathrm{Stab}(x)}{G/\mathrm{Stab}(x)} \arrow[d, "\varphi_g"] \\ \underset{(gh)*x}{\mathrm{Orb}(x)} \arrow[r, "f"] & \underset{(gh)\,\mathrm{Stab}(x)}{G/\mathrm{Stab}(x)} \end{tikzcd}$$

i.e. $f \circ \varphi_g = \varphi_g \circ f$.

We say
$$\mathrm{Orb}(x) \underset{G}{\approx} G/\mathrm{Stab}(x)$$

as G-sets

$G \curvearrowright X$ is ...

Transitive if $Orb(x) = X$

Simple if $Stab(x) = \{1\} \leq G$.

$G \curvearrowright X$ simply transitively

$$\Longrightarrow \quad X = Orb(x) \underset{G}{\approx} G/Stab(x) = G$$

$$X \underset{G}{\approx} G$$

eg. $D_n \curvearrowright$ labeled n-gons

eg. $Isom \curvearrowright \mathbb{R}^n$ transitively

with point stabilizer $O(n)$

$$\Longrightarrow \quad \mathbb{R}^n \approx Isom / O(n) \approx \mathbb{R}^n_{\#} \quad \checkmark$$

Application: Burnside

If $G \curvearrowright X$

$$\#\text{orbits} = |X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

eg. color faces of dodecahedron black or white. How many?

	class	# orbits of faces	# colorings fixed
	1	12	2^{12}
vertex	20	4	2^4
face	12	4	2^4
	12	4	2^4
edge	15	6	2^6

colorings =

$$\frac{1}{60}\left[2^{12} + 20 \cdot 2^4 + 24 \cdot 2^4 + 15 \cdot 2^6\right]$$

$$= \frac{1}{60}\left[2^{12} + 44 \cdot 2^4 + 15 \cdot 2^6\right]$$

$$= \frac{16}{60}\left[2^8 + 44 + 15 \cdot 2^2\right]$$

$$= \frac{16}{60}\left[256 + 44 + 60\right] = \frac{16}{60}\left[360\right]$$

$$= 16 \cdot 6 = 96$$

:)