HW 4 due <u>Wed</u> <u>Nov</u> 9.

From Exam 2:

Consider $H, K \leq G$ with $K \trianglelefteq G$.

Then $HK \leq G$ with $K \trianglelefteq HK$.

Define a map $\varphi : H \to HK/K$.
$$h \mapsto h/K.$$

hom: $\varphi(h_1 h_2) = (h_1 h_2)K$
$$= h_1 K \, h_2 K = \varphi(h_1)\varphi(h_2) \quad \checkmark$$

surj: Arbitrary element of HK/K looks like $(hk)K$ for some $h \in H$, $k \in K$.

Then $\varphi(h) = hK = (hk)K$ ✓.

kernel? Note:

$$\varphi(h) = 1K = K \iff h \in K.$$

$$\Longrightarrow \ker \varphi = H \cap K.$$

Conclusion: $H \cap K \trianglelefteq H$ and

$$\frac{H}{H \cap K} \cong \frac{HK}{K}.$$

"2nd Isomorphism Theorem"

Lagrange says

$$\frac{|H|}{|H \cap K|} = \frac{|HK|}{|H \cap K|}$$

$$\Longrightarrow |HK| = \frac{|H||K|}{|H \cap K|}$$ ■

Corollary:

$$|HK| = |H||K| \iff H \cap K = \{1\}$$

Have a <u>bijection</u> $HK \leftrightarrow H \times K$.

$$\begin{aligned}
h_1 k_1 = h_2 k_2 &\iff h_2^{-1} h_1 = k_2 k_1^{-1} \ (\in H \cap K) \\
&\iff h_2^{-1} h_1 = 1 = k_2 k_1^{-1} \\
&\iff h_1 = h_2 \ \& \ k_1 = k_2 \\
&\iff (h_1, k_1) = (h_2, k_2)
\end{aligned}$$

But <u>maybe</u> <u>not</u> $HK \approx H \times K$.
All we can say is

$$h_1 k_1 h_2 k_2 = \underbrace{h_1 h_2}_{\in H} \underbrace{(\overbrace{h_2^{-1} k_1 h_2}^{\in K \text{ because } K \trianglelefteq G}) k_2}_{\in K}.$$

So Define a funny product on $H \times K$:

$(h_1, k_1) \cdot (h_2, k_2) := (h_1 h_2, (h_2^{-1} k_1 h_2) k_2)$

Product is "direct" if $h_2^{-1} k_1 h_2 = k_1$

$k_1 h_2 = h_2 k_1$ $\forall h_1 \in H$

commute. $k_2 \in K$.

Otherwise consider the hom $\theta : H \to Aut(K)$ defined by $h \mapsto$ the map $\theta_h(k) = h^{-1} k h \in K$ (since $K \trianglelefteq G$).

Notation:

$$HK \approx H \rtimes_\theta K$$

semi-direct product with respect to $\theta : H \to Aut(K)$

"twisted product".

eg Let $id : H \to Aut(K)$

$h \mapsto$ identity map $\forall h$.

Then $H \rtimes_{id} K = H \times K$.

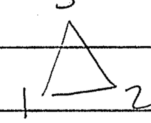
NOT TWISTED.

Do such things occur "in nature"?

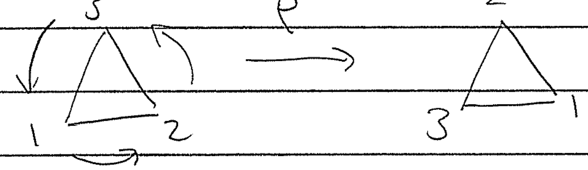
<u>yes!</u>

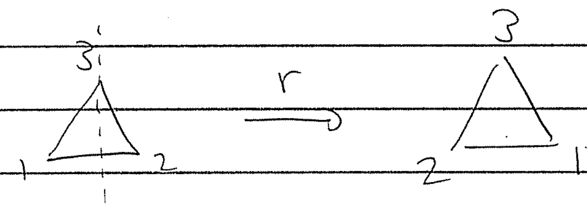
The Dihedral Group D_n

D_n = symmetries of a labeled regular n-gon.

e.g. D_3 acts on the triangle 

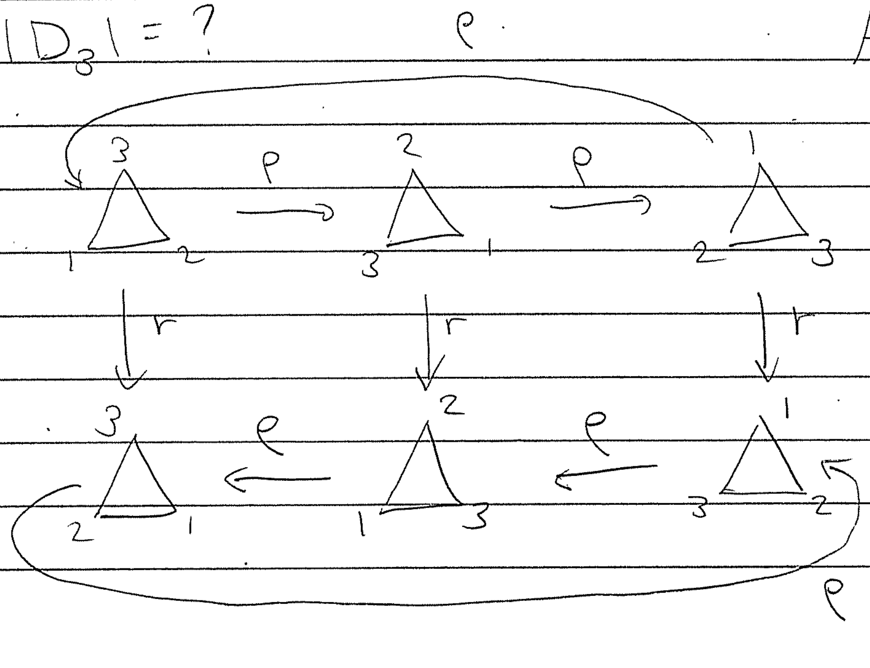
Two operations

Rotate: 

Reflect: 

Q: $|D_3| = ?$ A: $|D_3| = 6$

"Cayley Graph"



Q: $r\rho = \rho r$? NO.

$$\begin{tikzcd} \cdot \arrow[r, "\rho"] \arrow[d, "r"'] & \cdot \arrow[d, "r"] \\ \cdot & \cdot \arrow[l, "\rho"] \end{tikzcd}$$

means $\rho r \rho = r$
$r\rho = \rho^{-1} r$
$\neq \rho r$.
(since $\rho \neq \rho^{-1}$).

D_3 is the <u>smallest</u> non-abelian group.

Note: $D_3 = \langle r, \rho \rangle$
"generated" by r & ρ.

$= \{1, \rho, \rho^2, r, r\rho, r\rho^2\}$

Pic.

$$\begin{tikzcd} \rho^2 \arrow[r, leftrightarrow, "r"] \arrow[dd, bend right=60, "\rho"'] & r\rho^2 \arrow[d, "\rho"] \\ \rho \arrow[u, "\rho"] \arrow[r, leftrightarrow, "r"] & r\rho \arrow[d, "\rho"] \\ 1 \arrow[u, "\rho"] \arrow[r, leftrightarrow, "r"] & r \arrow[uu, bend right=60, "\rho"'] \end{tikzcd}$$

where's the triangle?

Arrow = left multiply by ρ or r $(r^{-1} = r)$
Columns = cosets $D_3/\langle \rho \rangle$
Rows = cosets $D_3/\langle r \rangle$

Note: $\langle r\rangle \cap \langle \rho\rangle = \{1\}$

$\Longrightarrow$ $|\langle r\rangle\langle \rho\rangle| = 2\cdot 3 = 6$

$\Longrightarrow$ $\langle r\rangle\langle \rho\rangle = D_3$

as <u>sets</u>

(group structure?)

Note: $\langle \rho\rangle \trianglelefteq D_3$

$\langle r\rangle$ <u>acts</u> on $\langle \rho\rangle$ by.

$1\rho^k 1^{-1} = \rho^k$ identity.

$$
\begin{aligned}
r\rho^k r^{-1} &= (r\rho r^{-1})(r\rho r^{-1})\cdots(r\rho r^{-1}) \\
&= \rho^{-1}\rho^{-1}\cdots\rho^{-1} \\
&= \rho^{-k}
\end{aligned}
$$

inversion

Say $\theta : \langle r\rangle \to \mathrm{Aut}(\langle \rho\rangle)$

$r \longmapsto$ inversion map.

$\Longrightarrow$

$$D_3 = \langle r\rangle \rtimes_\theta \langle \rho\rangle$$

$$\mathbb{Z}/2\mathbb{Z} \rtimes_\theta \mathbb{Z}/3\mathbb{Z}$$

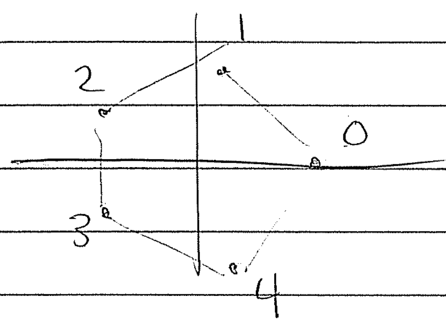
<u>twisted</u> product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.

Note: $D_3 \neq \underbrace{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}}_{\text{this is abelian}}$

(and $\approx \mathbb{Z}/6\mathbb{Z}$).
C.R.T.

In general, consider labeled regular n-gon in $\mathbb{R}^2$ with one vertex at $(1,0)$

eg.



maybe vertex $k = e^{\frac{2\pi i k}{n}}$.

Symmetries = D_n Dihedral group.

Generated by $r = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ reflect vertically.

$$\rho = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

Where $c = \cos\left(\frac{2\pi}{n}\right)$, $s = \sin\left(\frac{2\pi}{n}\right)$.

$$r\rho = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \begin{pmatrix} c & -s \\ -s & -c \end{pmatrix}$$

$$= \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \rho^{-1} r$$

$$\Longrightarrow \quad r \rho r = \rho^{-1}$$

$$D_3 = \langle r \rangle \ltimes \langle \rho \rangle$$
$$= \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z} \text{ non-abelian.}$$

$$\neq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \text{ . abelian!}$$

Recall: Matrices are conjugate if the <u>do</u> <u>the</u> <u>same</u> <u>thing</u> with respect <u>to</u> <u>different</u> <u>bases</u> . . .