Math 561 H | **Fall 2011**
Homework 5 | **Drew Armstrong**

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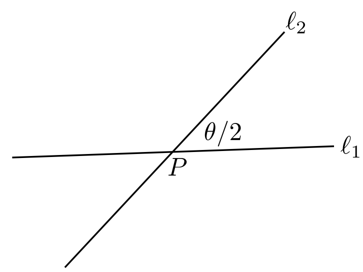
1. We saw in class that any element of the orthogonal group $O(2)$ has the form

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad F_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

The matrix R_θ (with determinant 1) **rotates** the plane around 0 counterclockwise by the angle θ. The matrix F_θ (with determinant -1) **reflects** the plane across the line through 0 that has angle $\theta/2$ measured counterclockwise from the x-axis.

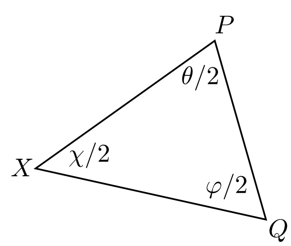
(a) For all angles $\alpha, \beta \in \mathbb{R}$, prove that $F_\alpha F_\beta = R_{\alpha-\beta}$.

(b) Consider lines ℓ_1 and ℓ_2 in $\mathbb{R}^2$ with intersection P and angle $\theta/2$ as below.



Let F_ℓ denote the reflection across line ℓ and let R^P_θ denote the rotation around the point P counterclockwise by θ. **Prove** that $F_{\ell_2} \circ F_{\ell_1} = R^P_\theta$. (Hint: You can assume that $P = 0$ and ℓ_1 is the x-axis. Use part (a).)

2. Consider the following triangle in $\mathbb{R}^2$.



Again let R^P_θ denote the rotation around point P counterclockwise by angle θ. **Prove** that

$$R^Q_\varphi \circ R^P_\theta = R^X_{-\chi}.$$

(Hint: Use Problem 1(b).) What happens when $\theta = \varphi \to 180°$?

3. Let $\mathrm{Isom}(\mathbb{R}^n)$ denote the group of isometries $\varphi : \mathbb{R}^n \to \mathbb{R}^n$. We know that if φ fixes the origin, then φ is an orthogonal linear map. Let $O(n) \leq \mathrm{Isom}(\mathbb{R}^n)$ denote the subgroup fixing the origin. Given $\alpha \in \mathbb{R}^n$, define the translation $t_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ by $t_\alpha(x) := x + \alpha$. Clearly this is an isometry. Let $\mathbb{R}^n_+ \leq \mathrm{Isom}(\mathbb{R}^n)$ denote the (abelian) subgroup of translations, which is isomorphic to vector addition on $\mathbb{R}^n$ via $t_\alpha \circ t_\beta = t_{\alpha+\beta}$.

(a) Prove that every isometry $f \in \mathrm{Isom}(\mathbb{R}^n)$ can be written **uniquely** in the form $f = t_\alpha \circ \varphi$ with $t_\alpha \in \mathbb{R}^n_+$ and $\varphi \in O(n)$. (Hint: Let $\alpha = f(0)$.)

(b) Given $\alpha \in \mathbb{R}^n$ and $\varphi \in O(n)$, prove that $\varphi \circ t_\alpha = t_{\alpha'} \circ \varphi$, where $\alpha' = \varphi(\alpha)$.

(c) Prove that $\mathbb{R}^n_+ \trianglelefteq \mathrm{Isom}(\mathbb{R}^n)$, and hence $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n_+ \rtimes O(n)$. (This is the prototypical example of a semi-direct product.) Describe how to multiply the elements $t_\alpha \circ \varphi$ and $t_\beta \circ \psi$. Conclude that $\mathrm{Isom}(\mathbb{R}^n) \not\approx \mathbb{R}^n_+ \times O(n)$.

4. The Lemma That Is Not Burnside's is a nice way to compute the number of orbits when a finite group G acts on a finite set S. Here you will prove it.

(a) Let $S^g = \{s \in S : gs = s\}$ be the set fixed by $g \in G$ and let $G_s = \{g \in G : gs = s\}$ be the subgroup of G that fixes $s \in S$. Count the elements of the set $\{(g, s) \in G \times S : gs = s\}$ in two different ways to show that

$$\sum_{g \in G} |S^g| = \sum_{s \in S} |G_s|.$$

(b) Let $G(s) = \{gs : g \in G\}$ be the orbit generated by $s \in S$ and let S/G denote the set of orbits (which, recall, partition the set S). Prove that

$$\sum_{s \in S} \frac{1}{|G(s)|} = |S/G|.$$

(c) Combine (a) and (b) to prove that

$$|S/G| = \frac{1}{|G|} \sum_{g \in G} |S^g|.$$

That is, the number of orbits is equal to the average number of elements of S fixed by an element of G. (Hint: Orbit-Stabilizer Theorem.)

5. We say a bracelet of size n is a circular string of n black and white beads. We say that two bracelets are **equal** if they differ by a dihedral symmetry. (You can rotate a bracelet and you can take it off your wrist, flip it over, and put it back on.) Use The Lemma That Is Not Burnside's to **compute the number of bracelets of size** 7. (Hint: The dihedral group D_7 acts on the set of 2^7 circular strings of 7 black and white beads, and the orbits are called bracelets. You know the conjugacy classes of D_7. How many strings are fixed by each conjugacy class?)