Math 561 H **Fall 2011**
Homework 4 **Drew Armstrong**

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1. Define the ring of quaternions $\mathbb{H} := \{a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a, b, c, d \in \mathbb{R}\}$, with the relations $\mathbf{1} = 1$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Define the quaternion absolute value by

$$|a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}|^2 := a^2 + b^2 + c^2 + d^2.$$

Note $\mathbb{H}$ is actually isomorphic to $\mathbb{R}^4$ as a vector space, but it has more structure than $\mathbb{R}^4$.

(a) Given $q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, define the quaternion conjugate $\bar{q} := a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ and show that $q\bar{q} = |q|^2$.

(b) Show that $\mathbb{H}$ is actually a division algebra by finding the inverse of $q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Note that $\mathbb{H}$ is not a field because it is not commutative.

(c) The nonzero quaternions $\mathbb{H}^\times$ are isomorphic to a subgroup of $GL_2(\mathbb{C})$ via the map

$$a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \leftrightarrow \begin{pmatrix} a + id & -b - ic \\ b - ic & a - id \end{pmatrix}.$$

Use this to prove that $|uv| = |u||v|$ for all $u, v \in \mathbb{H}$.

[The quaternions were discovered by William Rowan Hamilton on October 16, 1843, as he was walking with his wife along the Royal Canal in Dublin. To celebrate the discovery, he immediately carved this equation into the stone of the Brougham Bridge: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$.]

2. Recall that $\mathbb{Z}/n\mathbb{Z}$ has a unique (cyclic) subgroup of order d for each $d|n$. Use this to prove that $\sum_{d|n} \varphi(d) = n$, where φ is Euler's totient function. This formula can be used to compute φ recursively: $\varphi(n) = n - \sum_{d|n, d<n} \varphi(d)$.

3. Let G be a group and recall that its center is a normal subgroup $Z(G) \trianglelefteq G$. **Prove:** If $G/Z(G)$ is cyclic then G is abelian.

4. Explicitly describe the conjugacy classes of the Dihedral group

$$D_n := \langle r, \rho : r^2 = \rho^n = 1, \rho r = r\rho^{-1} \rangle.$$

Hint: Every element of D_n looks like $r\rho^k$ or ρ^k for some k.