Math 561 H **Fall 2011**
Homework 1 **Drew Armstrong**

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1. Let G be a group. Prove that the identity element of G is unique (that is, there is only one element of G satisfying the defining property of an identity element). This justifies our use of the special symbol "e".

2. Let G be a finite group.

(a) Show that there are an odd number of $x \in G$ such that $x^3 = e$.
(b) Show that there are an even number of $x \in G$ such that $x^2 \neq e$.

(Hint: What is the inverse of x^n?)

3. Let G be a finite group. For all $a, b \in G$, show that ab and ba have the same order as elements of G.

4. Let G be a group and fix an element $g \in G$. Define a function $\phi_g : G \to G$ by $\phi_g(h) = ghg^{-1}$ for all $h \in G$.

(a) Prove that ϕ_g is a bijection (one-to-one and onto).
(b) Prove that $\phi_g(ab) = \phi_g(a)\phi_g(b)$ for all $a, b \in G$.

These two properties mean that ϕ_g is an automorphism (a "symmetry") of G.

5. Suppose that $1, 9, 16, 22, 53, 74, 79, 81$ are eight members of a nine-element subgroup of $(\mathbb{Z}/91\mathbb{Z})^\times$. Which element has been left out? Recall: $(\mathbb{Z}/91\mathbb{Z})^\times$ is the multiplicative group of invertible elements of $\mathbb{Z}/91\mathbb{Z}$. What is the inverse of 9 in this group?

6. Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. We define the (real) general linear group, special linear group, orthogonal group, and special orthogonal group as follows:

$$\begin{aligned} GL_n(\mathbb{R}) &:= \{A \in M_n(\mathbb{R}) : \det A \neq 0\}, \\ SL_n(\mathbb{R}) &:= \{A \in M_n(\mathbb{R}) : \det A = 1\}, \\ O_n(\mathbb{R}) &:= \{A \in M_n(\mathbb{R}) : AA^T = I\}, \\ SO_n(\mathbb{R}) &:= \{A \in M_n(\mathbb{R}) : AA^T = I, \det A = 1\}. \end{aligned}$$

Why is $GL_n(\mathbb{R})$ a group? Show that $SL_n(\mathbb{R})$, $O_n(\mathbb{R})$ and $SO_n(\mathbb{R})$ are subgroups of $GL_n(\mathbb{R})$.

7. Given a matrix $A \in M_n(\mathbb{R})$ we can define a function from pairs $u, v \in \mathbb{R}^n$ of column vectors to scalars $\mathbb{R}$ by $(u, v)_A := u^T A v$. (Such a function is called a bilinear form — it is a generalization of the dot product.) If $(u, v)_A = (u, v)_B$ for all vectors u, v, **prove** that the matrices A, B are equal. (Hint: What if u, v are standard basis vectors?)

8. Let $(u, v) = u^T v$ denote the dot product of column vectors $u, v \in \mathbb{R}^n$. We define the length $\|u\|$ of a vector u by $\|u\|^2 := (u, u)$, so that $\|u - v\|$ represents the distance between two vectors u, v. Now consider an orthogonal matrix $A \in O_n(\mathbb{R})$. Given two vectors $u, v \in \mathbb{R}^n$, **prove** that $\|u - v\| = \|Au - Av\|$.

(It is also true — but harder to show — that **any** isometry (distance-preserving map) of $\mathbb{R}^n$ that fixes the origin $0 \in \mathbb{R}^n$ is given by $u \mapsto Au$ for some orthogonal matrix $A \in O_n(\mathbb{R})$. This fact is a bit surprising, and it gives a strong geometric meaning to the orthogonal group.)