HW 1 due Friday on Blackboard
before the lecture.

Recall from last time, complex L^2 space:

$$L^2[0,1] = \{ f : [0,1] \to \mathbb{C}, \int_0^1 |f(x)|^2 dx < \infty \}$$

This is Hermitian space:

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)^* g(x)\, dx$$

[$\alpha = a + ib \Rightarrow \alpha^* := \bar{\alpha} = a - ib$.
Later: A^* = conjugate transpose matrix.]

$$\langle f, g \rangle = \langle g, f \rangle^*$$

$$\langle f(x), g(x) + \alpha h(x) \rangle$$

$$= \int_0^1 f(x)^* (g(x) + \alpha h(x))\,dx$$

$$= \int_0^1 f(x)^* g(x) + \alpha \int_0^1 f(x)^* h(x)\,dx$$

$$= \langle f(x), g(x) \rangle + \alpha \langle f(x), h(x) \rangle$$

[Note $\langle \alpha f(x), g(x) \rangle = \alpha^* \langle f(x), g(x) \rangle$]

$$\langle f(x), f(x) \rangle = \int_0^1 f(x)^* f(x)\,dx$$

$$= \int_0^1 |f(x)|^2\,dx \geq 0.$$

$$= 0 \iff f(x) \equiv 0.$$

Fourier Series:

$$\chi_n(x) = e^{i2\pi n x}$$

Then $\ldots, \chi_{-2}(x), \chi_{-1}(x), \chi_0(x), \chi_1(x), \ldots$

is an orthonormal basis for $L^2[0,1]$.

(Indepence is easy ✓

Spanning is hard :(

[Theorem: $L^2[0,1]$ is "complete" w.r.t. norm $||f||^2 = \langle f(x), f(x) \rangle$ and every function in $L^2[0,1]$ has a convergent Fourier series.]

Expansion in this basis is called Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \chi_n(x)$$

[KEY: $\langle \chi_m(x), \chi_n(x) \rangle = \delta_{mn}$]

To compute Fourier coefficients:

$$\langle \chi_m(x), f(x) \rangle$$

$$= \langle \chi_m(x), \sum c_n \chi_n(x) \rangle$$

$$= \sum c_n \langle \chi_m(x), \chi_n(x) \rangle$$

$$= \sum c_n \delta_{mn}$$

$$= c_m$$

$$c_m = \langle \chi_m(x), f(x) \rangle$$

$$= \int_0^1 \chi_m(x)^* f(x)\, dx$$

$$= \int_0^1 e^{-i 2\pi m x} f(x)\, dx \, .$$

What about $L^2(\mathbb{R})$?

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C}, \int_{-\infty}^{\infty} |f(x)|^2 < \infty \right\}$$

Hermitian product:

$$\langle f(x), g(x) \rangle = \int_{-\infty}^{\infty} f(x)^* g(x)\, dx \, .$$

Problem: $L^2(\mathbb{R})$ does not have a "countable basis". Define

$$\chi_\xi(x) = e^{i2\pi\xi x} \quad \text{for any } \xi \in \mathbb{R}.$$

Then we have a Fourier <u>Transform</u>:

$$f(x) = \int_{-\infty}^{\infty} c(t)\, \chi_\xi(x)\, d\xi.$$

$$= \int_{-\infty}^{\infty} c(\xi)\, e^{i2\pi\xi x}\, d\xi$$

Function $c : \mathbb{R} \to \mathbb{C}$ describing the uncountably many coefficients is called the Fourier transform of $f(x)$.

Jargon: $\hat{f}(\xi) = c(\xi)$.

Coefficients can be computed as before:

$$c_m = \langle e^{i2\pi m x}, F(x) \rangle$$

$$\hat{f}(\xi) = \langle e^{i2\pi \xi x}, F(x) \rangle$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi \xi x} F(x)\, dx$$

One more example.

A random variable X is defined by a "density function"

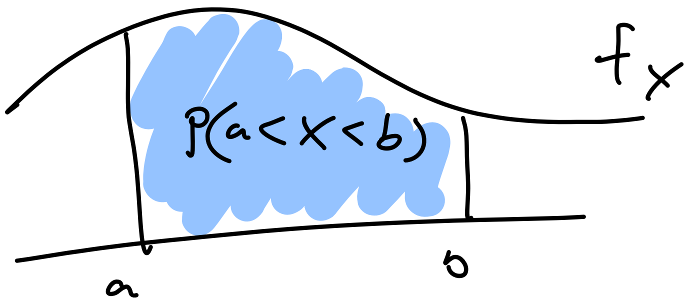
$$f_X : \mathbb{R} \to \mathbb{R}$$

satisfying

- $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$
- $\displaystyle\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

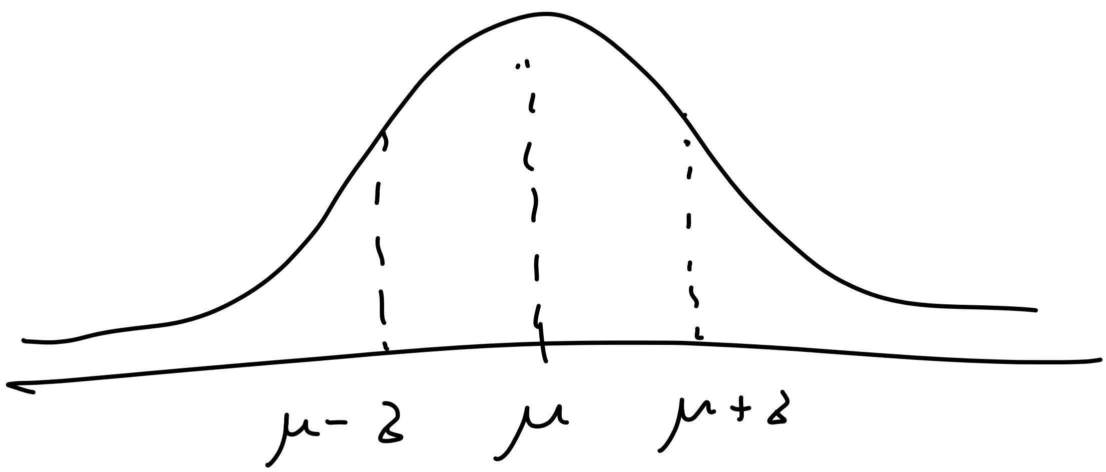
For any interval $a < b$, the probability that random X falls between a & b is

$$P(a < X < b) = \int_a^b f_X(x)\,dx$$



Example: Normal (Gaussian)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$



Simpler Example:

Let X be constant: $X = \alpha$.

What is the density function?

Need

$$P(a < X < b) = \int_a^b f_X(x)\,dx$$

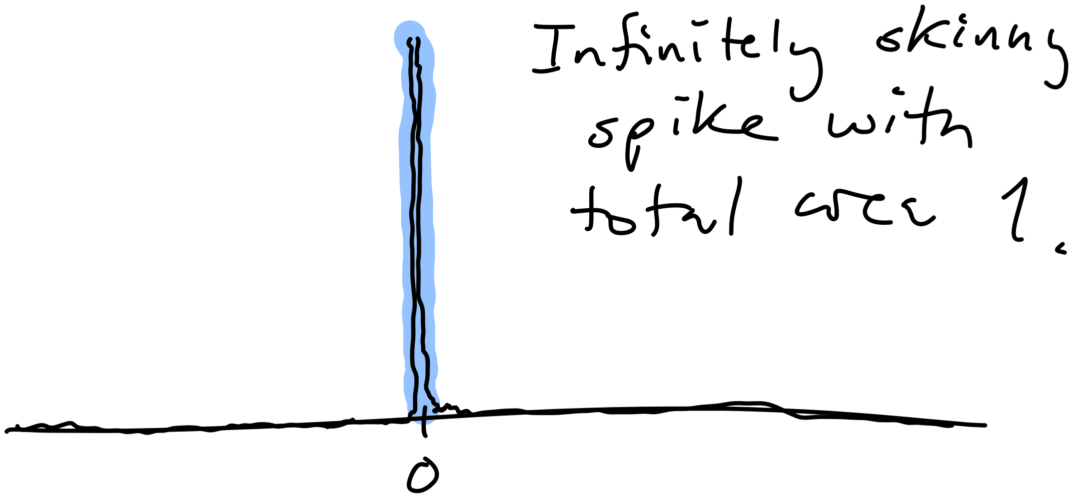
$$= \begin{cases} 1 & a < \alpha < b \\ 0 & \text{otherwise}. \end{cases}$$

There is no such function!

But there is a "fake function" called the Dirac delta function.

$$\int_a^b \delta(x)\,dx = \begin{cases} 1 & a < 0 < b \\ 0 & \text{else}. \end{cases}$$

Picture:



If $X = \alpha$ constant then

$$f_X(x) = \delta(x-\alpha).$$

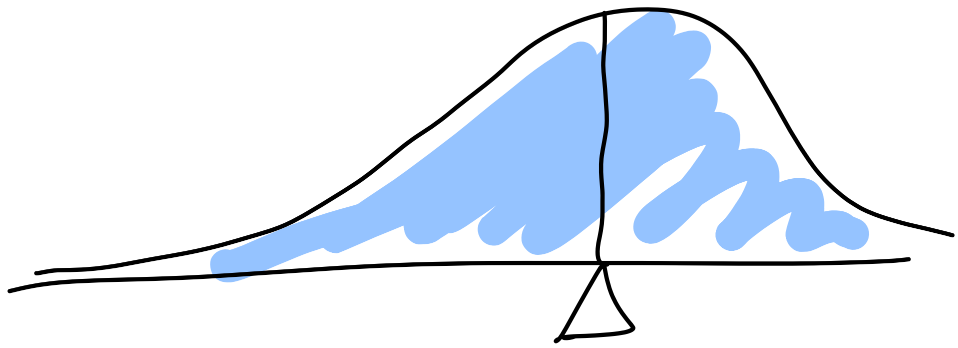
Idea: The Dirac δ-functions are a <u>basis</u> of $L^2(\mathbb{R})$:

$$f(x) = \int f(\xi)\,\delta(x-\xi)\,d\xi$$

Given r.v. X with density f_X define its expected value

$$E[X] = \int x\, f_X(x)\, dx$$

(Archimedes)



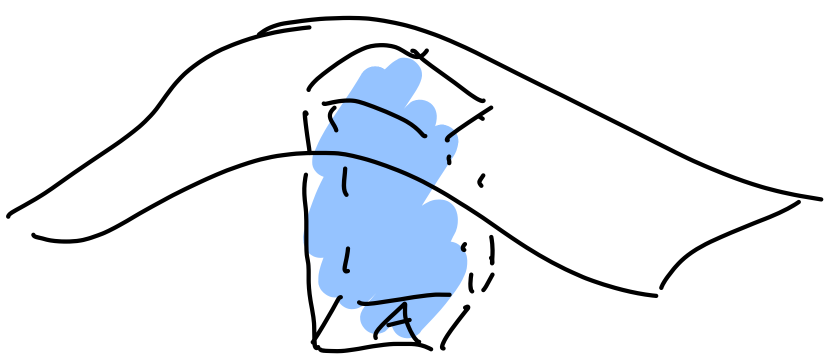
where does it balance?

Given two r.v. X, Y there is a joint density

$$f_{XY} : \mathbb{R}^2 \to \mathbb{R}$$

- $f_{XY}(x,y) \geq 0$

- $\iint_{\mathbb{R}^2} f_{XY}(x,y)\,dx\,dy = 1$



For any $A \subseteq \mathbb{R}^2$:

$$P((X,Y) \in A) = \iint_A f_{XY}(x,y)\,dx\,dy$$

Some formulas :

$$E[g(X)] = \int g(x) f_X(x)\,dx$$

$$E[g(X,Y)] = \iint g(x,y) f_{XY}(x,y)\,dx\,dy$$

Variance :

$$Var(X) = E[(X - E[X])^2]$$

$$= \int (x - E[X])^2 f_X(x)\,dx$$

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$

$$= \iint (x - E[X])(y - E[Y]) f_{XY}(x,y)\,dx\,dy$$

Why am I saying this ?

Because $\mathrm{Cov}(X, Y)$ is (almost) an inner product on the vector space of random variables.

Example: Flip coin n times with $p = P(\text{heads})$. Let $X =$ # times you get heads.

$$E[X] = \sum x \, P(X = x)$$

$$= 0 \, P(0 \text{ heads}) + 1 \, P(1 \text{ heads}) + \cdots + n \, P(n \text{ heads})$$

$$= np \quad [\text{Theorem}]$$

What about $E[X^2]$?

$E[X^2]$ = average value of $(\#\text{heads})^2$.

Formula:

$$E[g(x)] = \sum g(x) P(X=x)$$

$$E[X^2] = \sum x^2 P(X=x)$$

[Warning: $E[g(X)] \neq g(E[X])$!]

Variance:

$$E[(X - E[X])^2]$$

$$= \text{algebra}$$

$$= E[X^2] - E[X]^2$$

Covariance:

$$E[(X - E[X])(Y - E[Y])]$$

$$= \text{algebra} = E[XY] - E[X]E[Y].$$

Measures tendency of X & Y to move in same direction.

Properties:

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$$

$$\mathrm{Cov}(X, Y+\alpha Z) = \mathrm{Cov}(X,Y) + \alpha\,\mathrm{Cov}(X,Z)$$

$$\mathrm{Cov}(X,X) = \mathrm{Var}(X,X) \geq 0.$$

$$\mathrm{Var}(X) = 0 \implies X = 0 \quad ?$$

NOT QUITE:

$$\mathrm{Var}(X) = 0 \implies X = \text{constant}.$$

So Cov is positive-semidefinite, not positive-definite.