Exam 1 Friday in Ungar 411.

Office Hours Today & Tomorrow 3-4.

Today: Finish Proof of the Fundamental Theorem:

$$\dim R(A) = \dim C(A)$$
$$r_R = r_c$$

row rank column rank

Strategy:

We showed for invertible E, F:

$$R(EA) = R(A)$$
$$R(AF) \cong R(A).$$

So $\dim R(EA) = \dim(AF)$
$$= \dim(EAF)$$
$$= \dim R(A) = r_R$$

Similarly,

$$C(AF) = C(A)$$
$$C(EA) \cong C(A)$$

So $\dim C(AF) = \cdots = r_c$.

To complete to proof we will find inv. matrices E & F such that

$$EAF = m\left\{\begin{array}{c} \overbrace{\left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right)}^{n} \\ \underbrace{\hphantom{I_r}}_{r}\ \underbrace{\hphantom{OO}}_{n-r} \end{array}\right. \begin{array}{l} \}\, r \\ \}\, m-r \end{array}$$

Then it follows that

$$\begin{aligned} r_R &= \dim R(EAF) \\ &= r \\ &= \dim C(EAF) \\ &= r_c . \end{aligned}$$

How to find such matrices E & F?

Define "elementary matrices"
(elimination)

Three types:

$$D_i(\lambda) = \begin{array}{c c} & \begin{array}{ccccccc} & & & i & & & \end{array} \\ i & \begin{pmatrix} 1 & & & \vdots & & & \\ & \ddots & & \vdots & & & \\ & & 1 & \vdots & & & \\ \cdots & \cdots & \cdots & \lambda & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

$\lambda \neq 0$

$$L_{ij}(\lambda) = \begin{array}{c c} & \begin{array}{ccccccc} & & & i & & j & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} 1 & & & \vdots & & \vdots & \\ & \ddots & & \vdots & & \vdots & \\ & & 1 & \vdots & & \vdots & \\ \cdots & \cdots & \cdots & 1 & \cdots & \lambda & \\ & & & & \ddots & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix} \end{array}$$

$$T_{ij} = \begin{array}{c c} & \begin{array}{ccccccc} & & i & & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} 1 & & \vdots & & & \vdots & & \\ & \ddots & \vdots & & & \vdots & & \\ & 1 & \vdots & & & \vdots & & \\ \cdots & \cdots & 0 & 1 & \cdots & 1 & & \\ & & \vdots & & \ddots & \vdots & & \\ & & \vdots & & 1 & \vdots & & \\ \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \end{array}$$

Observe that elem matrices are

invertible:

$$D_i(\lambda)^{-1} = D_i(1/\lambda) \qquad \lambda \neq 0$$

$$L_{ij}(\lambda)^{-1} = L_{ij}(-\lambda)$$

$$T_{ij}^{-1} = T_{ij}$$

Can use elem. mats. to perform "row elimination" & "col elimination".

Recall:

(ith row EA) is a linear combo. of rows of A. Exact:

$$(\text{ith row } E) = (e_{i1}, e_{i2} \cdots e_{in})$$

$$(\text{ith row } A) = \vec{a}_i^{\,T}$$

$$(\text{ith row } EA) = (\text{ith row } E)\, A$$

$$= (e_{i1} \cdots e_{in}) \begin{pmatrix} - \vec{a}_1^{\,T} - \\ \vdots \\ - \vec{a}_n^{\,T} - \end{pmatrix}$$

$$= e_{i1}\vec{a}_1^T + e_{i2}\vec{a}_2^T + \cdots + e_{in}\vec{a}_n^T \quad \checkmark$$

Now let E = elementary.

ith row $D_j(\lambda)A = \begin{cases} \vec{a}_i^T & j \neq i, \\ \lambda \vec{a}_j^T & j = i. \end{cases}$

ith row $L_{jk}(\lambda)A = \begin{cases} \vec{a}_i^T & j \neq i, \\ \vec{a}_j^T + \lambda \vec{a}_k & j = i. \end{cases}$

ith row $T_{jk}A = \begin{cases} \vec{a}_i & i \neq j, k, \\ \vec{a}_j & i = k, \\ \vec{a}_k & i = j. \end{cases}$

Similar formulas for

$\underbrace{AE}_{\text{performs a column operation on } A.}$, E elementary.

Finally, the reduction algorithm.

- IF $A = 0$ stop.
- Swap rows/cols to get nonzero entry in top left.
- Scale 1st row (or col) to make top left entry $= 1$:

$$\left(\begin{array}{c|cccc} 1 & * & \cdots & * & * \\ \hline * & & & & \\ \vdots & & & & \\ * & & & & \end{array}\right)$$

- Apply L_{ij} matrices to left & right to eliminate entries in 1st row/col:

$$\left(\begin{array}{c|cccc} 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & & & & \\ \vdots & & A' & & \\ 0 & & & & \end{array}\right)$$

- Repeat on smaller matrix A'.

Stop when you hit a matrix of zeros:

$$E_k \cdots E_1 A F_1 \cdots F_\ell$$

$$= \left(\begin{array}{c|c} \begin{matrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)$$

QED.

Remark: If $A \in \mathbb{Z}^{m \times n}$ then a similar algorithm over $\mathbb{Z}$ produces (much harder)

$$EAF = \left(\begin{array}{c|c} \begin{matrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)$$

where $d_1, \ldots, d_r \in \mathbb{Z}$ satisfy

$$d_1 \mid d_2 \mid \cdots \mid d_r.$$

Called the "Smith Normal Form".

Recall: Solving a linear system

$$A\vec{x} = \vec{b}$$

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

is related to finding a left inverse for A: $CA = I$.

$$A\vec{x} = \vec{b}$$

$$CA\vec{x} = C\vec{b}$$

$$\vec{x} = C\vec{b}.$$

In fact we can take C as a product of elementary row ops:

$$E_1(A \mid \vec{b})$$

$$(E_1 A \mid E_1\vec{b})$$

$$\rightsquigarrow E_2(E_1 A \mid E_1\vec{b})$$

maybe don't look at this right now. We'll discuss later.

$$\left(E_2E_1A \mid E_2E_1\vec{b}\right)$$

$$\rightsquigarrow \cdots$$

$$\left(I \mid \underbrace{E_k\cdots E_2E_1\vec{b}}_{\text{solution.}}\right).$$

Apply Fund. Thm. to matrix inversion. Note: Linear system

$A\vec{x}=\vec{b}$ has a solution

$\iff \vec{b}\in C(A)$.

Indeed,

$$A\vec{x} = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \underbrace{x_1\vec{a}_1 + \cdots + x_n\vec{a}_n}_{\substack{\text{linear combo} \\ \text{of cols of } A}}$$

More generally, matrix eqn

$$\underset{m \times n}{A} \; \underset{n \times m}{X} = \underset{m \times m}{I_m}$$

$$A \begin{pmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_m \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{e}_1 & \cdots & \vec{e}_m \\ | & & | \end{pmatrix}$$

$$\begin{pmatrix} | & & | \\ A\vec{x}_1 & \cdots & A\vec{x}_m \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{e}_1 & \cdots & \vec{e}_m \\ | & & | \end{pmatrix}$$

has a solution

$\iff \vec{e}_i \in C(A)$ for all i.

$\iff C(A) = \mathbb{R}^m$

$\iff r_c = m$

(full column-rank)

Similarly, matrix A has a left inverse

$XA = I_n \iff R(A) = \mathbb{R}^n$

$\iff r_R = n$

(full row-rank)

By the Fund. Thm. (A $m \times n$)

A has right inverse

$\iff r_C = m$

$\iff r = m$.

A has left inverse

$\iff r_R = n$

$\iff r = n$.

A has both left & right inverse

$\iff m = r = n$.

(A is a square matrix of full rank)