Exam Friday in person, Ungar 411.

I will test understanding of <u>definitions</u> and <u>small proofs</u>.

e.g. Manipulations with inner products: Say over $\mathbb{R}$.

$$\langle \vec{u} - t\vec{v}, \vec{u} - t\vec{v} \rangle$$

$$= \langle \vec{u}, \vec{u} - t\vec{v} \rangle - t\langle \vec{v}, \vec{u} - t\vec{v} \rangle$$

Bilinearity in first position.

$$= \langle \vec{u}, \vec{u} \rangle - t\langle \vec{u}, \vec{v} \rangle$$
$$- t\langle \vec{v}, \vec{u} \rangle + t^2\langle \vec{v}, \vec{v} \rangle.$$

Bilinearity in second position

$$= \langle \vec{u}, \vec{u} \rangle - 2t\langle \vec{u}, \vec{v} \rangle + t^2\langle \vec{v}, \vec{v} \rangle.$$

KNOW FROM AXIOMS:

- $\langle \vec{w}, \vec{w} \rangle \geq 0$

- $\langle \vec{w}, \vec{w} \rangle = 0 \iff \vec{w} = \vec{0}$.

$$0 \leq \langle \vec{u} - t\vec{v}, \vec{u} - t\vec{v} \rangle \qquad \vec{w} = \vec{u} - t\vec{v}$$

$$= \langle \vec{u}, \vec{u} \rangle - 2t \langle \vec{u}, \vec{v} \rangle + t^2 \langle \vec{v}, \vec{v} \rangle$$

True for any vectors $\vec{u}, \vec{v} \in V$ & any scalar $t \in \mathbb{R}$.

After this we used a trick to pick a very special value of t:

$$t = \langle \vec{u}, \vec{v} \rangle / \langle \vec{v}, \vec{v} \rangle.$$

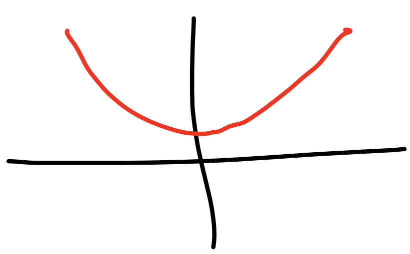
$\rightsquigarrow$ Cauchy-Schwarz.

Another trick: Think of

$$\langle \vec{u} - t\vec{v}, \vec{u} - t\vec{v} \rangle = f(t)$$

as real function of t.

Parabola



Since $f(t) \geq 0$ does not have two real roots.

Discriminant ≤ 0

$$0 \geq b^2 - 4ac = \left(-2\langle \vec{u}, \vec{v} \rangle\right)^2 - 4\langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle.$$

$$0 \geq 4\langle \vec{u}, \vec{v} \rangle^2 - 4\langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle.$$

$$\langle \vec{u}, \vec{v} \rangle^2 \leq \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle.$$

KNOW various interpretations of matrix multiplication.

$A = (a_{ij})$, $B = (b_{ij})$

$AB = (c_{ij})$.

$$c_{ij} = \sum_{k=1}^{?} a_{ik} b_{kj}$$

Generally not a useful way to think!

A rows $\vec{a}_i^T$

B cols $\vec{b}_j$

Then

ij entry AB = ith row A $\cdot$ jth col B

$$= \underbrace{\vec{a}_i^T \vec{b}_j}_{\text{dot product.}}$$

Application: Square $A \in \mathbb{R}^{n \times n}$.

A has columns $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^n$

A^T has rows $\vec{a}_1^T, \dots, \vec{a}_n^T$.

ij entry of $A^TA = \underbrace{\vec{a}_i^T \vec{a}_j}_{\text{dot product.}}$

ij entry of $I = \delta_{ij}$

$$= \begin{cases} 1 & i=j \\ 0 & i \neq j. \end{cases}$$

Therefore.

$$A^T A = I \iff \vec{a}_i^T \vec{a}_j = \delta_{ij}$$

$\iff$ cols $\vec{a}_i$ are orthonormal.

Another point of view:

A $\quad l \times m$ $\quad$ cols $\vec{a}_1, \dots, \vec{a}_m \in \mathbb{R}^l$

B $\quad m \times n$ $\quad$ rows $\vec{b}_1^T, \dots, \vec{b}_m \in \mathbb{R}^n$

Then

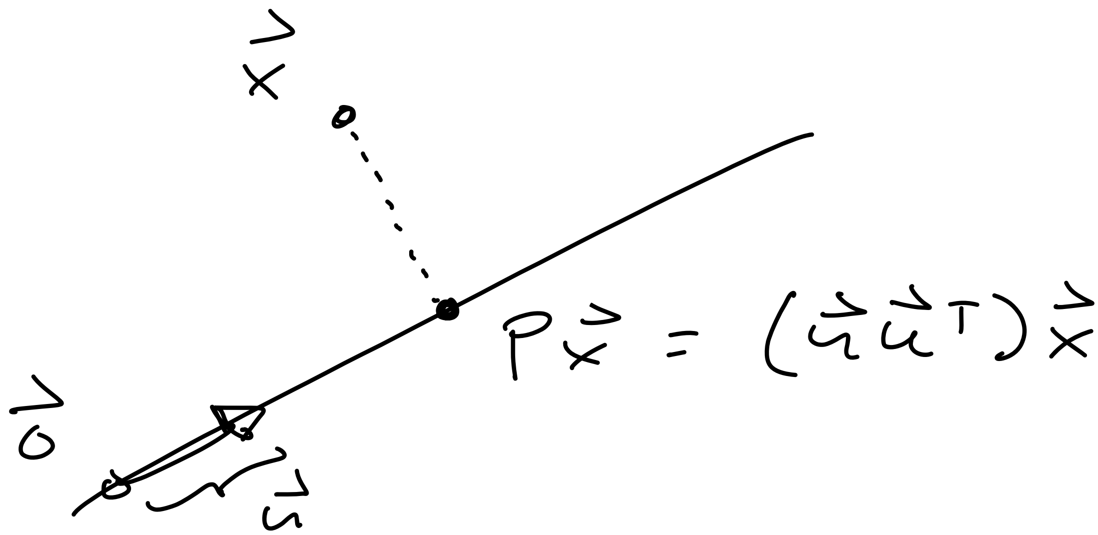
$$AB = \sum_{i=1}^{m} \underbrace{\vec{a}_i \vec{b}_i^T}$$

each of these
is an $l \times n$ matrix.

Example: Projection matrix.

$P = \vec{u}\,\vec{u}^T$

$=$ projection onto line spanned by unit vector $\vec{u}$.



e.g. $\vec{u} = (\cos t, \sin t) \in \mathbb{R}^2$

$$P = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \begin{pmatrix} \cos t & \sin t \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2 t & \cos t \sin t \\ \cos t \sin t & \sin^2 t \end{pmatrix}.$$

Another point of view:

jth col AB $=$ A (jth col B).

Example: What 2×2 matrix sends $(1,0) \to (-1,2)$
$(0,1) \to (3,-4)$?

Know $AI = A$

jth col A

$=$ jth col AI

$=$ A (jth col I)

$= A\vec{e}_j$

1st col A $= A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$

2nd col A $= A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\-4\end{pmatrix}$

$$\Longrightarrow \quad A = \begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix} \quad \checkmark$$

More:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a + 0b \\ c + 0d \end{pmatrix}$$

$$= \begin{pmatrix} a \\ c \end{pmatrix}.$$

Slightly more advanced:

Each row of AB is a linear combo of rows of B.

ith row AB = (ith row A) B.

NAMES:

$A = (a_{ij})$

ith row A = $(a_{i1} \; a_{i2} \; \cdots \; a_{im})$

$$B = \begin{pmatrix} — \vec{b}_1^{\,T} — \\ \vdots \\ — \vec{b}_m^{\,T} — \end{pmatrix}$$

Then ith row of AB

$$= \underbrace{\begin{pmatrix} a_{i1} & \cdots & a_{im} \end{pmatrix}}_{1 \times m} \underbrace{\begin{pmatrix} — \vec{b}_1^{\,T} — \\ \vdots \\ — \vec{b}_m^{\,T} — \end{pmatrix}}_{m \times n.}$$

$$= \underbrace{a_{i1} \vec{b}_1^{\,T} + a_{i2} \vec{b}_2^{\,T} + \cdots + a_{im} \vec{b}_m^{\,T}}_{\text{a linear combo of rows of } B.}$$

Abstract properties of matrix arithmetic (don't involve any discussion of matrix entries):

e.g. $(AB)^* = B^* A^*$

e.g. Say A^{-1} exists then

claim $(A^*)^{-1}$ exists & is

equal to $(A^{-1})^*$

Proof: Assume A^{-1} exists.

Want to show $A^*(A^{-1})^* = I$

& $(A^{-1})^* A^* = I$.

Indeed, we have

$$A^*(A^{-1})^* = (A^{-1}A)^*$$

$$= I^*$$

$$= I \checkmark$$

Other direction similar.

KEY Definition:

$A^{-1} = B$ means $AB = I$

&

$BA = I$

More . . .

Say A is orthogonal if

$$A^{-1} = A^T.$$

If A & B are orthogonal, show that AB is orthogonal.

Proof: Assume $A^{-1} = A^T$, $B^{-1} = B^T$.

Want $(AB)^{-1} = (AB)^T$.

exercise: prove this

Indeed:

$$\begin{aligned} (AB)^{-1} &= B^{-1}A^{-1} \\ &= B^T A^T \\ &= (AB)^T \quad \checkmark \end{aligned}$$

Interesting consequence:

If A & B have orthonormal cols then AB has orthonormal cols.